The Exact Partially Symmetric Rank of Products of W-States

Sketch of proof

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The known upper bound and our aim. For $k \geq 2$ and $d_1, \ldots, d_k \geq 3$, Canino, Casarotti and Santarsiero [1, Theorem 1.1] give an explicit decomposition proving the **general upper bound** for the partially symmetric rank

$$R_{\mathbf{d}}\left(\prod_{i=1}^{k} x_i^{d_i-1} y_i\right) \leq M := 2^{k-1}\left(\sum_i d_i - 2k + 2\right).$$

Every term of such a decomposition has the form $\prod_i (a_i x_i + b_i y_i)^{d_i}$. We prove the matching lower bound $r \geq M$. The idea is simple to state: turn an arbitrary decomposition into moment conditions, build M evaluation vectors, and show that they are linearly independent because a weighted Gram matrix is invertible. Together with the upper bound this gives the exact rank, answering TR-29 of the *Open Problems in Numerical Linear Algebra* catalogue [2].

1. A change of coordinates

We study

$$F(x, y) = \prod_{i=1}^{k} x_i^{d_i-1} y_i, \qquad k \geq 2, \quad d_i \geq 3,$$

and suppose it has an r-term decomposition

$$F(x, y) = \sum_{\ell=1}^{r} c_\ell \prod_{i=1}^{k} (a_{i\ell} x_i + b_{i\ell} y_i)^{d_i}. \tag{1}$$

We want to show that every such decomposition needs many terms.

After discarding zero terms, choose for each i a number γ_i with $a_{i\ell} + \gamma_i b_{i\ell} \neq 0$ for all ℓ; each term excludes at most one value, so such a choice exists. Make the invertible change of coordinates

$$x_i = X_i, \qquad y_i = \gamma_i X_i + Y_i,$$

and put

$$z_{i\ell} = \frac{b_{i\ell}}{a_{i\ell} + \gamma_i b_{i\ell}}, \qquad w_\ell = c_\ell \prod_i (a_{i\ell} + \gamma_i b_{i\ell})^{d_i}.$$

Then (1) becomes

$$\boxed{\prod_{i=1}^{k} X_i^{d_i-1}(\gamma_i X_i + Y_i) = \sum_{\ell=1}^{r} w_\ell \prod_{i=1}^{k} (X_i + z_{i\ell} Y_i)^{d_i}.} \tag{2}$$

Each pair $(a_{i\ell}, b_{i\ell})$ is normalised to one ratio $z_{i\ell}$, and the scale is absorbed into the weight w_ℓ. Arrange the parameters as

$$Z_c = \begin{pmatrix} z_{11} & \cdots & z_{1r} \\ \vdots & & \vdots \\ z_{k1} & \cdots & z_{kr} \end{pmatrix}, \qquad z_\ell = (z_{1\ell}, \ldots, z_{k\ell}) \in \mathbb{C}^k.$$

Column ℓ is one point. So an r-term decomposition has become r **weighted points**. These points are given to us by the decomposition; what we choose later are the functions used to test them.

2. Comparing coefficients: moment conditions

Compare the coefficients of $\prod_i X_i^{d_i-e_i} Y_i^{e_i}$ on the two sides of (2) and divide by $\prod_i \binom{d_i}{e_i}$:

$$\boxed{\sum_{\ell=1}^{r} w_\ell \prod_i z_{i\ell}^{e_i} = \prod_i m_i(e_i), \qquad 0 \le e_i \le d_i,} \tag{3}$$

where

$$m_i(e) = \begin{cases} \gamma_i, & e = 0, \\ 1/d_i, & e = 1, \\ 0, & 2 \le e \le d_i. \end{cases}$$

This family of moment conditions is equivalent to (2): it matches every coefficient of the two polynomials. Nothing is lost; the identity is simply rewritten in terms of points and weights.

3. Generalized moment conditions

Let $Z = (Z_1, \dots, Z_k)$ be the variables of test polynomials, put

$$\mathcal{P}_{\mathbf{d}} = \{P \in \mathbb{C}[Z_1, \dots, Z_k] : \deg_{Z_i} P \le d_i \text{ for all } i\},$$

and define the weighted evaluation

$$\boxed{L(P) := \sum_{\ell=1}^{r} w_\ell P(z_\ell).} \tag{4}$$

If $P(Z) = \sum_e p_e \prod_i Z_i^{e_i} \in \mathcal{P}_{\mathbf{d}}$, the moment conditions are equivalent to

$$\boxed{L(P) = \sum_e p_e \prod_i m_i(e_i) \qquad \text{for all } P \in \mathcal{P}_{\mathbf{d}}.} \tag{5}$$

We call (5) the **generalized moment conditions**. They follow from the monomial ones by linearity, and taking $P = \prod_i Z_i^{e_i}$ recovers each original condition. Hence

$$\text{identity (2)} \iff \text{moment conditions (3)} \iff \text{generalized moment conditions (5)}.$$

In particular, if $P(Z) = \prod_i P_i(Z_i) \in \mathcal{P}_{\mathbf{d}}$, then

$$L(P) = \prod_i L_i(P_i), \qquad L_i(Z_i^e) = m_i(e). \tag{6}$$

4. Building the Gram matrix

The generalized moment conditions let us evaluate L on any $P \in \mathcal{P}_{\mathbf{d}}$. We use the particular expressions

$$\boxed{P_{ab}(Z) := H_t(Z)\phi_a(Z)\phi_b(Z), \qquad a, b = 1, \dots, M,} \tag{7}$$

where

$$H_t(Z) = \prod_{i=1}^{k} (1 + t_i Z_i)^{d_i},$$

the test functions $\phi_1, \dots, \phi_M$ are given in Step 6, and the auxiliary parameters $t = (t_1, \dots, t_k)$ satisfy $1 + t_i z_{i\ell} \neq 0$ for all i, ℓ. Define the **Gram coefficients**

$$\boxed{\mathcal{G}_{ab} := L(P_{ab}) = \sum_{\ell=1}^{r} w_\ell H_t(z_\ell)\phi_a(z_\ell)\phi_b(z_\ell).} \tag{8}$$

When $P_{ab} \in \mathcal{P}_{\mathbf{d}}$ this coefficient is computed directly from (5). When P_{ab} still has denominators, $L(P_{ab})$ is still a finite weighted sum, but it needs a separate argument.

Collecting all coefficients gives

$$\boxed{\mathcal{G} = (\mathcal{G}_{ab})_{a,b=1}^{M} = E^{\mathsf{T}} W E,} \tag{9}$$

with

$$\boxed{E = \begin{pmatrix} \phi_1(z_1) & \cdots & \phi_M(z_1) \\ \vdots & & \vdots \\ \phi_1(z_r) & \cdots & \phi_M(z_r) \end{pmatrix} \in \mathbb{C}^{r \times M},} \qquad W = \operatorname{diag}\big(w_1 H_t(z_1), \dots, w_r H_t(z_r)\big). \tag{10}$$

5. Invertibility of the Gram matrix gives the bound

So $\mathcal{G}$ is the weighted bilinear Gram matrix of the evaluation vectors. If it is invertible, then

$$\boxed{\mathcal{G} \text{ invertible} \Longrightarrow E \text{ has full column rank} \Longrightarrow M \leq r.} \tag{11}$$

Indeed, if $Ec = 0$ then $\mathcal{G}c = E^{\mathsf{T}} W E c = 0$, so $c = 0$ and the M columns of E are independent. We use the transpose, not the conjugate transpose, so neither W nor $\mathcal{G}$ needs to be positive definite. It remains to choose the test functions and to find one admissible t for which $\mathcal{G}$ is invertible.

6. The test functions and the nondegenerate Gram matrix

Let ι denote a multi-index with entries in $\{0,1\}$, and write $h_i(Z_i) = 1 + t_i Z_i$. The test functions come in batches.

Basic batch $\mathscr{B}_0$:

$$\prod_{j=1}^{k} \frac{Z_j^{\iota_j}}{h_j(Z_j)}, \qquad \iota \in \{0,1\}^k,$$

with 2^k functions.

Batch $\mathscr{B}_i$ $(i = 1, \dots, k)$:

$$\frac{Z_i^s}{h_i(Z_i)^s} \prod_{j \neq i} \frac{Z_j^{\iota_j}}{h_j(Z_j)}, \qquad 2 \le s \le d_i - 1, \quad \iota \in \{0,1\}^{k-1},$$

with $(d_i - 2)2^{k-1}$ functions. Listing $\mathscr{B}_0, \mathscr{B}_1, \dots, \mathscr{B}_k$ in order and calling them $\phi_1, \dots, \phi_M$, the total is

$$\boxed{M = 2^k + \sum_i (d_i - 2)2^{k-1} = 2^{k-1}\left(\sum_i d_i - 2k + 2\right).} \tag{12}$$

The two functions ϕ_a, ϕ_b in (7) are any two entries of this single list. For instance, if both come from $\mathscr{B}_0$, with indices ι, ι', then $P_{ab}(Z) = \prod_i Z_i^{\iota_i + \iota'_i}(1 + t_i Z_i)^{d_i - 2} \in \mathcal{P}_{\mathbf{d}}$, so (5) applies directly.

Why $\mathcal{G}$ is invertible. Write $\gamma_i = m_i(0)$ and $\beta_i = m_i(1) = 1/d_i$. Three facts finish the proof.

- *Block diagonal form.* Pairing $\mathscr{B}_i$ with any other batch produces, in coordinate i, a polynomial whose degrees lie between 2 and d_i, so its moment vanishes. Hence $\mathcal{G} = \operatorname{diag}(\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_k)$, and

$$\mathcal{G}_0 = \bigotimes_{i=1}^{k} G_i(t_i), \qquad G_i(t_i) = \begin{pmatrix} \gamma_i + (d_i - 2)t_i\beta_i & \beta_i \\ \beta_i & 0 \end{pmatrix}, \qquad \det G_i = -\beta_i^2 \neq 0.$$

- *Anti-triangular blocks.* Inside $\mathcal{G}_i$, the (s, s') block vanishes when $s + s' \le d_i$ and equals one fixed $2^{k-1} \times 2^{k-1}$ matrix $H_i(t)$ when $s + s' = d_i + 1$. Reversing block columns gives a triangular matrix, so $\det \mathcal{G}_i = \pm \det H_i(t)^{d_i - 2}$.

- *One good parameter.* Expanding at $t_i = \infty$, $H_i(t) = (-1)^{d_i - 1}\beta_i t_i^{-d_i} \bigotimes_{j \neq i} G_j(t_j) + O(t_i^{-d_i - 1})$, whose leading coefficient is invertible. So each $\det H_i$ is a nonzero rational function, and a single finite t avoiding all their zeros and poles and all $1 + t_i z_{i\ell} = 0$ makes every block invertible at once.

Example: $k = 3$, $d_1 = d_2 = d_3 = 3$. The list splits as

$$\underbrace{\phi_1, \dots, \phi_8}_{\mathscr{B}_0}, \quad \underbrace{\phi_9, \dots, \phi_{12}}_{\mathscr{B}_1}, \quad \underbrace{\phi_{13}, \dots, \phi_{16}}_{\mathscr{B}_2}, \quad \underbrace{\phi_{17}, \dots, \phi_{20}}_{\mathscr{B}_3},$$

and $\mathcal{G} = \operatorname{diag}(G_1 \otimes G_2 \otimes G_3, H_1, H_2, H_3)$ is a 20×20 invertible matrix. Hence $r \ge 20$, and the rank of $x_1^2 y_1 x_2^2 y_2 x_3^2 y_3$ is exactly 20. In general $r \ge M$, which matches the upper bound and gives the exact partially symmetric rank.

References

[1] S. Canino, A. Casarotti, P. Santarsiero. *A new bound on the rank of tensor product of W-states.* arXiv:2512.05828v1 (2025), Theorem 1.1.

[2] Open Problems in Numerical Linear Algebra, problem TR-29. https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/tensor-computations/TR-29.