The Exact Partially Symmetric Rank of Products of W-States

A proof via moment conditions and weighted Gram matrices

Wenqi Zhu* Ping Nie†

November 2026

Abstract

We study the minimum number of products of powers of binary linear forms needed to express $\prod_{i=1}^{k} x_i^{d_i-1}y_i$. For $k \geq 2$ and $d_i \geq 3$, we prove that this number is $2^{k-1}(\sum_i d_i - 2k + 2)$, matching the upper bound of Canino, Casarotti and Santarsiero. The lower bound starts from an arbitrary complex decomposition. A change of coordinates converts its coefficients into weighted points satisfying product moment conditions. We construct rational test functions whose evaluation vectors have a block diagonal weighted Gram matrix. Its blocks are nondegenerate for a common finite choice of auxiliary parameters, so the number of decomposition terms is at least the number of test functions. The argument allows repeated points and zero coordinates and uses no positivity assumption.

1 Introduction and main result

For $\mathbf{d} = (d_1, \ldots, d_k)$, consider the multihomogeneous polynomial

$$F(x, y) = \prod_{i=1}^{k} x_i^{d_i-1}y_i. \tag{1}$$

The pair (x_i, y_i) represents the ith symmetric tensor block, of order d_i. Thus F represents a product of k generalized W-states, up to a nonzero normalization scalar. In this polynomial description, a rank-one term with respect to the prescribed blocks is a product $\prod_i (a_i x_i + b_i y_i)^{d_i}$.

Definition 1.1 (Partially symmetric rank)**.** The partially symmetric rank $R_{\mathbf{d}}(F)$ is the least integer r for which there are complex coefficients $c_\ell, a_{i\ell}, b_{i\ell}$ satisfying

$$F(x, y) = \sum_{\ell=1}^{r} c_\ell \prod_{i=1}^{k} (a_{i\ell}x_i + b_{i\ell}y_i)^{d_i}. \tag{2}$$

The blocks (x_i, y_i) and their degrees d_i are fixed throughout.

Canino, Casarotti and Santarsiero [1, Theorem 1.1] give an explicit decomposition proving the general upper bound

$$R_{\mathbf{d}}(F) \leq M, \qquad M := 2^{k-1}\left(\sum_{i=1}^{k} d_i - 2k + 2\right), \qquad k \geq 2, \quad d_i \geq 3. \tag{3}$$

Our contribution is the matching lower bound for every decomposition of the form (2), without any restriction on its support. This is the partially symmetric rank question in problem TR-29 of the Open Problems in Numerical Linear Algebra catalogue [2].

Theorem 1.1 (Exact partially symmetric rank)**.** *Let $k \geq 2$ and $d_i \geq 3$ for every i. Over $\mathbb{C}$,*

$$R_{\mathbf{d}}\left(\prod_{i=1}^{k} x_i^{d_i-1}y_i\right) = 2^{k-1}\left(\sum_{i=1}^{k} d_i - 2k + 2\right). \tag{4}$$

*Corresponding author. Mathematical Institute, University of Oxford, Andrew Wiles Building, Woodstock Road, Oxford OX2 6GG, United Kingdom. wenqi.zhu@maths.ox.ac.uk

†David R. Cheriton School of Computer Science, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada. ping.nie@uwaterloo.ca

The proof of the lower bound is developed in Sections 2–7 and completed at the end of Section 7. The upper bound is (3).

Example 1.1 (Three cubic factors)**.** For $F = x_1^2y_1x_2^2y_2x_3^2y_3$, the theorem gives $R_{(3,3,3)}(F) = 20$. More generally, $R_{(3,\dots,3)}(F) = 2^{k-1}(k+2)$.

The proof follows a concrete sequence. An arbitrary r-term decomposition gives r weighted points after a change of coordinates. Coefficient matching determines their moments. We then construct M test functions and prove that their evaluation vectors are linearly independent by showing that a weighted Gram matrix is invertible. Consequently $r \geq M$. The test functions and the matrix blocks are given explicitly in Section 7; the longer coefficient calculations are collected in the appendices.

All ranks in Theorem 1.1 refer to the prescribed partially symmetric blocks. The argument does not assert the same lower bound for ordinary tensor rank or border rank. All pairings below are complex bilinear, using transpose rather than conjugate transpose.

2 A change of coordinates

Fix any decomposition (2), deleting zero summands. We first replace each pair of coefficients $(a_{i\ell}, b_{i\ell})$ by one affine parameter, absorbing its scale into the weight of the summand.

Lemma 2.1 (Affine normalization)**.** *There exist $\gamma_1, \dots, \gamma_k \in \mathbb{C}$ such that the invertible coordinate changes $x_i = X_i$, $y_i = \gamma_i X_i + Y_i$ transform (2) into the equivalent identity*

$$\prod_{i=1}^{k} X_i^{d_i-1}(\gamma_i X_i + Y_i) = \sum_{\ell=1}^{r} w_\ell \prod_{i=1}^{k} (X_i + z_{i\ell}Y_i)^{d_i}, \tag{5}$$

where every w_ℓ is nonzero and

$$z_{i\ell} = \frac{b_{i\ell}}{a_{i\ell} + \gamma_i b_{i\ell}}, \qquad w_\ell = c_\ell \prod_{i=1}^{k} (a_{i\ell} + \gamma_i b_{i\ell})^{d_i}. \tag{6}$$

Proof. For each i, choose γ_i so that $a_{i\ell} + \gamma_i b_{i\ell} \neq 0$ for every ℓ. Such a choice is possible: each nonzero factor excludes at most one value of γ_i. After the coordinate change, $a_{i\ell}x_i + b_{i\ell}y_i = (a_{i\ell} + \gamma_i b_{i\ell})(X_i + z_{i\ell}Y_i)$. Substitution gives (5) and (6). □

Arrange the parameters into the matrix

$$Z_c = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1r} \\ z_{21} & z_{22} & \cdots & z_{2r} \\ \vdots & \vdots & & \vdots \\ z_{k1} & z_{k2} & \cdots & z_{kr} \end{pmatrix} \in \mathbb{C}^{k\times r}, \qquad z_\ell = (z_{1\ell}, \dots, z_{k\ell}) \in \mathbb{C}^k. \tag{7}$$

Its ℓth column specifies the point z_ℓ. Thus the original decomposition determines r points with weights w_ℓ. These points are fixed in the subsequent argument; we choose functions to evaluate at them. Repeated points and zero coordinates are allowed. The symbol $Z = (Z_1, \dots, Z_k)$ will denote formal variables, distinct from the parameter matrix Z_c.

3 Coefficient matching and moment conditions

Lemma 3.1 (Moment conditions)**.** *The normalized identity (5) is equivalent to*

$$\sum_{\ell=1}^{r} w_\ell \prod_{i=1}^{k} z_{i\ell}^{e_i} = \prod_{i=1}^{k} m_i(e_i), \qquad 0 \leq e_i \leq d_i, \tag{8}$$

where

$$m_i(e) = \begin{cases} \gamma_i, & e = 0, \\ \beta_i := 1/d_i, & e = 1, \\ 0, & 2 \leq e \leq d_i. \end{cases} \tag{9}$$

Proof. Compare the coefficients of $\prod_i X_i^{d_i-e_i}Y_i^{e_i}$ in (5) and divide by $\prod_i \binom{d_i}{e_i}$. The coefficient contributed by the ith factor on the left is γ_i when $e_i = 0$, is 1 when $e_i = 1$, and is zero otherwise. This gives (8)–(9). Conversely, these identities match every coefficient of the two multihomogeneous polynomials, so they imply (5). □

The term *moment* refers to the weighted monomial evaluations on the left of (8). The weights are complex; they are not assumed positive or normalized to sum to one. In particular, the moment conditions retain all the information in the normalized polynomial identity, rather than providing only necessary tests.

4 Generalized moment conditions

Let $\mathcal{P}_{\mathbf{d}} := \{P \in \mathbb{C}[Z_1, \dots, Z_k] : \deg_{Z_i} P \le d_i \text{ for all } i\}$. Define the weighted evaluation functional

$$L(P) := \sum_{\ell=1}^{r} w_\ell P(z_\ell), \qquad P \in \mathcal{P}_{\mathbf{d}}. \tag{10}$$

For each i, let L_i be the linear functional on univariate polynomials of degree at most d_i determined by $L_i(Z_i^e) = m_i(e)$.

Lemma 4.1 (Generalized moment conditions)**.** *The moment conditions* (8) *are equivalent to*

$$L(P) = \sum_e p_e \prod_{i=1}^{k} m_i(e_i) \quad \text{for every } P(Z) = \sum_e p_e \prod_{i=1}^{k} Z_i^{e_i} \in \mathcal{P}_{\mathbf{d}}. \tag{11}$$

In particular, for $P(Z) = \prod_i P_i(Z_i)$ *with* $\deg P_i \le d_i$,

$$L\left(\prod_{i=1}^{k} P_i(Z_i)\right) = \prod_{i=1}^{k} L_i(P_i). \tag{12}$$

Proof. Expand P in the monomial basis and apply (8) term by term. Conversely, taking $P(Z) = \prod_i Z_i^{e_i}$ in (11) recovers each moment condition. Expanding the univariate factors gives (12). □

Thus the normalized decomposition, the monomial moment conditions, and the generalized moment conditions are equivalent descriptions of the same identity. Formula (11) determines $L(P)$ from the target polynomial whenever P lies in the specified degree range. We also use $L(R) = \sum_\ell w_\ell R(z_\ell)$ for rational functions R defined at every point, but (11) does not directly apply to such functions.

5 Constructing the weighted Gram matrix

We now select a family of expressions to which weighted evaluation will be applied. Let $t = (t_1, \dots, t_k) \in \mathbb{C}^k$ satisfy $1 + t_i z_{i\ell} \neq 0$ for every i, ℓ, and set

$$h_i(Z_i) := 1 + t_i Z_i, \qquad H_t(Z) := \prod_{i=1}^{k} h_i(Z_i)^{d_i}. \tag{13}$$

Choose M rational test functions $\phi_1, \dots, \phi_M$ defined at the points z_ℓ. Their specific choice is given in Section 7. For each $a, b = 1, \dots, M$, put

$$P_{ab}(Z) := H_t(Z)\phi_a(Z)\phi_b(Z), \tag{14}$$

and define the Gram coefficient

$$\mathcal{G}_{ab} := L(P_{ab}) = \sum_{\ell=1}^{r} w_\ell H_t(z_\ell)\phi_a(z_\ell)\phi_b(z_\ell). \tag{15}$$

If $P_{ab} \in \mathcal{P}_{\mathbf{d}}$, this coefficient can be evaluated directly by (11). The remaining entries will be handled by an exact rational identity in Appendix B.

Lemma 5.1 (Gram factorization). *The matrix $\mathcal{G} = (\mathcal{G}_{ab})_{a,b=1}^{M}$ satisfies*

$$\mathcal{G} = E^{\mathsf{T}} W E, \tag{16}$$

where

$$E = \begin{pmatrix} \phi_1(z_1) & \phi_2(z_1) & \cdots & \phi_M(z_1) \\ \phi_1(z_2) & \phi_2(z_2) & \cdots & \phi_M(z_2) \\ \vdots & \vdots & & \vdots \\ \phi_1(z_r) & \phi_2(z_r) & \cdots & \phi_M(z_r) \end{pmatrix} \in \mathbb{C}^{r\times M}, \tag{17}$$

and

$$W = \begin{pmatrix} w_1 H_t(z_1) & 0 & \cdots & 0 \\ 0 & w_2 H_t(z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_r H_t(z_r) \end{pmatrix} \in \mathbb{C}^{r\times r}. \tag{18}$$

Proof. The (a, b) entry of $E^{\mathsf{T}} W E$ is exactly the sum in (15). □

Each column of E is the evaluation vector $\mathrm{ev}(\phi_a) = (\phi_a(z_1), \ldots, \phi_a(z_r))^{\mathsf{T}}$. The indices a and b select two functions from the same list. They do not refer to two different types of test.

6 From Gram invertibility to a rank lower bound

The matrix $\mathcal{G}$ is the weighted bilinear Gram matrix of the evaluation vectors. Its invertibility gives the required lower bound on the number of decomposition terms.

Lemma 6.1 (The dimension bound). *For the matrices in Lemma 5.1,*

$$\mathcal{G} \text{ invertible} \quad \Longrightarrow \quad \operatorname{rank} E = M \quad \Longrightarrow \quad r \geq M. \tag{19}$$

Proof. If $Ec = 0$, then $\mathcal{G}c = E^{\mathsf{T}} W E c = 0$. Invertibility of $\mathcal{G}$ gives $c = 0$, so the M columns of E are linearly independent in $\mathbb{C}^r$. □

This implication requires neither a positive weight matrix nor a Hermitian inner product. It remains to choose the M test functions and prove that their Gram matrix is invertible for at least one admissible finite parameter vector t.

7 A specific family of test functions

We now fix the test functions. All remaining assertions concern this specific choice. Write $[k] = \{1, \ldots, k\}$, and use ι for a multi-index with entries in $\{0, 1\}$.

7.1 The function list and its evaluation matrix

The basic family is

$$\mathscr{B}_0 := \left\{ \prod_{j=1}^{k} \frac{Z_j^{\iota_j}}{h_j(Z_j)} : \iota \in \{0,1\}^k \right\}. \tag{20}$$

For each $i \in [k]$, add the family

$$\mathscr{B}_i := \left\{ \frac{Z_i^s}{h_i(Z_i)^s} \prod_{j\neq i} \frac{Z_j^{\iota_j}}{h_j(Z_j)} : 2 \leq s \leq d_i - 1, \quad \iota \in \{0,1\}^{[k]\setminus\{i\}} \right\}. \tag{21}$$

List $\mathscr{B}_0, \mathscr{B}_1, \ldots, \mathscr{B}_k$ in this order. Within $\mathscr{B}_i$, order first by increasing s, then lexicographically by ι; order $\mathscr{B}_0$ lexicographically. Denote the resulting list by $\phi_1, \ldots, \phi_M$. Its length is

$$M = 2^k + \sum_{i=1}^{k} (d_i - 2) 2^{k-1} = 2^{k-1} \left(\sum_{i=1}^{k} d_i - 2k + 2 \right). \tag{22}$$

This is a count of the indexed list; its linear independence is proved below. Both ϕ_a and ϕ_b in (14) are chosen from this list, possibly from the same family.

Example 7.1 (The twenty functions for three cubic factors)**.** When $k = 3$ and $d_1 = d_2 = d_3 = 3$, let $h_i = 1+t_iZ_i$ and $h = h_1h_2h_3$. With the above ordering, the four rows below list the four families:

$$\begin{aligned}(\phi_1, \dots, \phi_8) &= \frac{1}{h}(1, Z_3, Z_2, Z_2Z_3, Z_1, Z_1Z_3, Z_1Z_2, Z_1Z_2Z_3), \\ (\phi_9, \dots, \phi_{12}) &= \frac{Z_1^2}{h_1^2h_2h_3}(1, Z_3, Z_2, Z_2Z_3), \\ (\phi_{13}, \dots, \phi_{16}) &= \frac{Z_2^2}{h_1h_2^2h_3}(1, Z_3, Z_1, Z_1Z_3), \\ (\phi_{17}, \dots, \phi_{20}) &= \frac{Z_3^2}{h_1h_2h_3^2}(1, Z_2, Z_1, Z_1Z_2).\end{aligned} \tag{23}$$

Their evaluations form

$$E = \big[E_0 \mid E_1 \mid E_2 \mid E_3\big], \qquad E_0 \in \mathbb{C}^{r\times 8}, \quad E_i \in \mathbb{C}^{r\times 4} \ (i = 1, 2, 3). \tag{24}$$

For example, the ℓth row of E_1 is $\frac{z_{1\ell}^2}{h_1(z_{1\ell})^2h_2(z_{2\ell})h_3(z_{3\ell})}(1, z_{3\ell}, z_{2\ell}, z_{2\ell}z_{3\ell})$.

7.2 The basic block and the vanishing cross blocks

Partition the Gram matrix according to the function families. Let $\mathcal{G}_0$ be its block on $\mathscr{B}_0$, and let $\mathcal{G}_i$ be its block on $\mathscr{B}_i$ for $i \geq 1$.

Lemma 7.1 (Block diagonal structure)**.** *For every admissible t, the Gram matrix of (20)–(21) has the form*

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_0 & 0 & \cdots & 0 \\ 0 & \mathcal{G}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{G}_k \end{pmatrix}, \qquad \mathcal{G}_0 = \bigotimes_{i=1}^{k} G_i(t_i), \tag{25}$$

where

$$G_i(t_i) = \begin{pmatrix} \gamma_i + (d_i - 2)t_i\beta_i & \beta_i \\ \beta_i & 0 \end{pmatrix}, \qquad \det G_i = -\beta_i^2 \neq 0. \tag{26}$$

In particular, $\mathcal{G}_0$ is invertible for every admissible t.

Proof. For two basic functions with indices $\iota, \jmath$, multiplication by H_t cancels the denominators and gives

$$P_{ab}(Z) = \prod_i Z_i^{\iota_i+\jmath_i}h_i(Z_i)^{d_i-2} \in \mathcal{P}_{\mathbf{d}}. \tag{27}$$

The product rule (12) shows that the Gram entry factors over the coordinates. For each coordinate the three possibilities $\iota_i + \jmath_i = 0, 1, 2$ give, respectively, $\gamma_i + (d_i - 2)t_i\beta_i$, β_i, and 0. This proves the Kronecker product formula and (26).

For a function from $\mathscr{B}_i$ paired with one from $\mathscr{B}_0$, the ith coordinate factor of P_{ab} is $Z_i^{s+\jmath_i}h_i(Z_i)^{d_i-s-1}$. It is a polynomial whose monomials have degrees between 2 and d_i, so its L_i-value is zero. All other factors lie within their prescribed degree bounds. The same factor occurs when a function from $\mathscr{B}_i$ is paired with one from $\mathscr{B}_j$, $j \neq i$. Hence all cross blocks vanish. The full entry formulas are recorded in Appendix A. □

Example 7.2 (An explicit basic Gram block)**.** For $k = 2$, write $A_i = \gamma_i + (d_i - 2)t_i\beta_i$. In the order $(0,0), (0,1), (1,0), (1,1)$,

$$G_1 \otimes G_2 = \begin{pmatrix} A_1A_2 & A_1\beta_2 & \beta_1A_2 & \beta_1\beta_2 \\ A_1\beta_2 & 0 & \beta_1\beta_2 & 0 \\ \beta_1A_2 & \beta_1\beta_2 & 0 & 0 \\ \beta_1\beta_2 & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

The product structure comes from (12), even though the points z_ℓ need not form a Cartesian grid.

7.3 The additional blocks

Put $q = 2^{k-1}$. Inside $\mathcal{G}_i$, group the functions by $s = 2, \dots, d_i - 1$. Each (s, s') block is a $q \times q$ matrix whose rows and columns are indexed by $\iota, \jmath$ on $[k] \setminus \{i\}$. Define the $q \times q$ matrix $H_i(t)$ by

$$[H_i(t)]_{\iota,\jmath} = \sum_{\ell=1}^{r} w_\ell \frac{z_{i\ell}^{d_i+1}}{1 + t_i z_{i\ell}} \prod_{j \neq i} z_{j\ell}^{\iota_j + \jmath_j} (1 + t_j z_{j\ell})^{d_j - 2}. \tag{29}$$

Here H_i is a matrix; it is distinct from the scalar polynomial H_t in (13).

Lemma 7.2 (The anti-triangular blocks)**.** *The (s, s') block of $\mathcal{G}_i$ is zero if $s + s' \leq d_i$ and equals $H_i(t)$ if $s + s' = d_i + 1$. Consequently,*

$$\det \mathcal{G}_i = \pm \det H_i(t)^{d_i - 2}. \tag{30}$$

Proof. The relevant entry is

$$\sum_{\ell=1}^{r} w_\ell z_{i\ell}^{s+s'} (1 + t_i z_{i\ell})^{d_i - s - s'} \prod_{j \neq i} z_{j\ell}^{\iota_j + \jmath_j} (1 + t_j z_{j\ell})^{d_j - 2}. \tag{31}$$

If $s + s' \leq d_i$, the expression inside the weighted sum is a polynomial in $\mathcal{P}_{\mathbf{d}}$ whose ith-coordinate factor has all degrees at least 4. Its moment therefore vanishes. If $s + s' = d_i + 1$, the entry is (29). Reversing the order of the s'-block columns gives a block lower triangular matrix with $d_i - 2$ copies of H_i on the diagonal. This proves (30). □

Example 7.3 (Column reversal when $d_i = 5$)**.** There are three groups, indexed by $s = 2, 3, 4$. Writing J_i, K_i for the remaining $q \times q$ blocks, the matrices before and after block-column reversal are

$$\mathcal{G}_i = \begin{pmatrix} 0 & 0 & H_i \\ 0 & H_i & J_i \\ H_i & J_i & K_i \end{pmatrix} \quad \longmapsto \quad \begin{pmatrix} H_i & 0 & 0 \\ J_i & H_i & 0 \\ K_i & J_i & H_i \end{pmatrix}. \tag{32}$$

Thus only the invertibility of H_i is needed; no formula for J_i or K_i is required.

7.4 A common finite parameter choice

The remaining task is to show that all the matrices H_i can be made invertible at the same admissible parameter vector.

Lemma 7.3 (Nonzero determinant of each additional block)**.** *For every i, $\det H_i(t)$ is a nonzero rational function of t. More precisely, with the other parameters fixed, its matrix has the Laurent expansion at $t_i = \infty$*

$$H_i(t) = (-1)^{d_i - 1} \beta_i t_i^{-d_i} \bigotimes_{j \neq i} G_j(t_j) + O(t_i^{-d_i - 1}). \tag{33}$$

The leading coefficient matrix is invertible.

Proof. Expand $z^{d_i+1}/(1 + t_i z)$ through order $t_i^{-d_i}$ in (29). The coefficients involving $z_i^{d_i}, \dots, z_i^2$ vanish by the moment conditions. The next coefficient involves z_i; the product rule gives $\beta_i \bigotimes_{j \neq i} G_j(t_j)$, with the sign in (33). Appendix B gives the exact finite identity and verifies this calculation also when some $z_{i\ell} = 0$. Each G_j is invertible by Lemma 7.1, so the leading coefficient is invertible and $\det H_i$ is not identically zero. □

Proposition 7.1 (Simultaneous Gram nondegeneracy)**.** *For any points and weights satisfying (8), there exists a finite $t \in \mathbb{C}^k$ such that $1 + t_i z_{i\ell} \neq 0$ for every i, ℓ and the Gram matrix of (20)–(21) is invertible.*

Proof. Write each nonzero rational function $\det H_i = N_i / D_i$ with nonzero polynomials N_i, D_i. The polynomial

$$Q(t) := \left(\prod_{i,\ell} (1 + t_i z_{i\ell}) \right) \prod_{i=1}^{k} N_i(t) D_i(t) \tag{34}$$

is nonzero. Since $\mathbb{C}$ is infinite, there is a finite parameter vector with $Q(t) \neq 0$. At this vector all evaluations are defined and all H_i are invertible. By Lemma 7.2, each $\mathcal{G}_i$ is invertible; by Lemma 7.1, the full block diagonal matrix $\mathcal{G}$ is invertible. □

The one-parameter expansions in Lemma 7.3 prove that the individual determinants are nonzero rational functions. They are not used to take all parameters to infinity simultaneously. The finite choice in (34) supplies the single parameter vector needed in the dimension argument.

Example 7.4 (The 20×20 Gram matrix)**.** For the cubic triple in Example 7.1, each additional family has only $s = 2$, so $\mathcal{G}_i = H_i$ for $i = 1, 2, 3$. The full matrix is

$$\mathcal{G} = \begin{pmatrix} G_1 \otimes G_2 \otimes G_3 & 0_{8\times 4} & 0_{8\times 4} & 0_{8\times 4} \\ 0_{4\times 8} & H_1 & 0_{4\times 4} & 0_{4\times 4} \\ 0_{4\times 8} & 0_{4\times 4} & H_2 & 0_{4\times 4} \\ 0_{4\times 8} & 0_{4\times 4} & 0_{4\times 4} & H_3 \end{pmatrix}. \tag{35}$$

Here $G_i = \left(\begin{smallmatrix} \gamma_i + t_i/3 & 1/3 \\ 1/3 & 0 \end{smallmatrix}\right)$ and each H_i is 4×4. For example, $H_1 = \frac{1}{3} t_1^{-3} (G_2 \otimes G_3) + O(t_1^{-4})$ with t_2, t_3 fixed. The other two blocks have the analogous expansions. Proposition 7.1 makes all four diagonal blocks invertible at one finite t. Thus the twenty columns of E are independent.

Proof of Theorem 1.1. Take an arbitrary decomposition (2) with r nonzero terms. By Lemmas 2.1 and 3.1, it determines points and nonzero weights satisfying (8) with $\beta_i = 1/d_i$. Construct the M test functions (20)–(21) and choose t using Proposition 7.1. Their Gram matrix is invertible, so Lemma 6.1 yields $r \geq M$. This applies to every decomposition, hence $R_{\mathbf{d}}(F) \geq M$. Combining with (3) proves (4). □

A Coordinate factors in the Gram entries

This appendix records the cancellation and degree checks used in Lemma 7.1. For a basic function indexed by $\jmath$ and a function from $\mathscr{B}_i$ indexed by (s, ι), their product with H_t is

$$P_{ab}(Z) = Z_i^{s+\jmath_i} h_i(Z_i)^{d_i - s - 1} \prod_{j \neq i} Z_j^{\iota_j + \jmath_j} h_j(Z_j)^{d_j - 2}. \tag{36}$$

The exponent $d_i - s - 1$ is nonnegative because $s \leq d_i - 1$. The degrees in coordinate i range from $s + \jmath_i \geq 2$ to $d_i - 1 + \jmath_i \leq d_i$. Thus the ith factor has zero L_i-value. Each other factor has degree at most d_j. Applying (12) gives $L(P_{ab}) = 0$.

Next choose functions from $\mathscr{B}_i$ and $\mathscr{B}_j$, with $i \neq j$. Let their indices be (s, ι) and $(s', \jmath)$, respectively. Then

$$\begin{aligned} P_{ab}(Z) = {} & Z_i^{s+\jmath_i} h_i(Z_i)^{d_i - s - 1} Z_j^{s' + \iota_j} h_j(Z_j)^{d_j - s' - 1} \\ & \times \prod_{m \neq i,j} Z_m^{\iota_m + \jmath_m} h_m(Z_m)^{d_m - 2}. \end{aligned} \tag{37}$$

Both distinguished coordinate factors have nonnegative powers of h_i, h_j and monomial degrees between 2 and their respective degree bounds. All remaining factors are also within their bounds. The product rule therefore gives $L(P_{ab}) = 0$.

Finally, for two functions from the same family $\mathscr{B}_i$, the coordinate factors are

$$P_{ab}(Z) = Z_i^{s+s'} h_i(Z_i)^{d_i - s - s'} \prod_{j \neq i} Z_j^{\iota_j + \jmath_j} h_j(Z_j)^{d_j - 2}. \tag{38}$$

This is polynomial when $s + s' \leq d_i$, yielding the zero blocks in Lemma 7.2. At $s + s' = d_i + 1$, one denominator remains, yielding H_i in (29). In particular, this last case is not an application of the polynomial moment formula to a rational function; it requires the calculation below.

B The leading coefficient of the rational blocks

We give the exact identity underlying Lemma 7.3. This also shows explicitly why zero coordinates of the decomposition points cause no difficulty.

Lemma B.1 (Finite division identity)**.** *For an integer $d \geq 1$, $t \neq 0$, and $1 + tz \neq 0$,*

$$\frac{z^{d+1}}{1 + tz} = \sum_{a=0}^{d-1} \frac{(-1)^a z^{d-a}}{t^{a+1}} + \frac{(-1)^d z}{t^d (1 + tz)}. \tag{39}$$

For each fixed $z \in \mathbb{C}$, the final term is $O(t^{-d-1})$ as $t \to \infty$; when $z = 0$, every term is identically zero.

Proof. Multiply by $1+tz$. The consecutive terms in the finite sum cancel, leaving z^{d+1} after the remainder is included. For $z \neq 0$, $z/(1+tz) = O(t^{-1})$; for $z = 0$ it is zero. □

Detailed proof of Lemma 7.3. Fix i and treat t_i as the variable, keeping the other parameters fixed. For row and column indices $\iota, \jmath$, set $A_{\iota,\jmath}(Z_{-i}) = \prod_{j\neq i} Z_j^{\iota_j+\jmath_j}(1+t_jZ_j)^{d_j-2}$. Each coordinate degree of $A_{\iota,\jmath}$ is at most d_j. Substituting (39) with $d = d_i$ into (29) gives

$$[H_i(t)]_{\iota,\jmath} = \sum_{a=0}^{d_i-1} (-1)^a t_i^{-a-1} L\Big(Z_i^{d_i-a} A_{\iota,\jmath}(Z_{-i})\Big) + O(t_i^{-d_i-1}). \tag{40}$$

The remainder estimate is valid because the sum over the fixed points is finite. The finite identity applies also to points with $z_{i\ell} = 0$; no division by such a coordinate is used.

For $a = 0, \dots, d_i - 2$, the ith exponent $d_i - a$ lies between 2 and d_i. The corresponding L-value is zero by (12). For $a = d_i - 1$, the same rule gives

$$L(Z_iA_{\iota,\jmath}(Z_{-i})) = \beta_i \prod_{j\neq i} L_j\Big(Z_j^{\iota_j+\jmath_j}(1+t_jZ_j)^{d_j-2}\Big) = \beta_i \left[\bigotimes_{j\neq i} G_j(t_j)\right]_{\iota,\jmath}. \tag{41}$$

Equations (40) and (41) prove (33).

For completeness, put $C_i = (-1)^{d_i-1}\beta_i \bigotimes_{j\neq i} G_j(t_j)$. This $q \times q$ matrix is invertible. Taking determinants gives

$$\det H_i(t) = t_i^{-d_iq}\left(\det C_i + O(t_i^{-1})\right), \qquad q = 2^{k-1}, \quad \det C_i \neq 0. \tag{42}$$

Thus $\det H_i$ cannot be the zero rational function. Equivalently, the calculation can be carried out as a formal Laurent expansion over the field $\mathbb{C}(t_j : j \neq i)$. □

Statements and Declarations

Funding. The research of Wenqi Zhu was supported by the Engineering and Physical Sciences Research Council under grant EP/Y028872/1, *Mathematical Foundations of Intelligence: An "Erlangen Programme" for AI*. Ping Nie gratefully acknowledges the support of AI4ALPHA and NetMind.AI.

Competing interests. The authors have no competing interests to declare that are relevant to the content of this article.

Data availability. The exact verification scripts, certificates and Lean 4 files supporting the finite computations are available from the corresponding author on reasonable request.

Use of generative AI and AI-assisted technologies. During the preparation of this work, the authors used AI tools (Claude, Anthropic; Codex, OpenAI; Qwen APP, Alibaba) to generate candidate constructions and code, perform computations, draft mathematical arguments, and improve the readability of the text. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication. The lower-bound argument is a complete written proof; its finite coefficient identities were additionally checked in exact arithmetic, and a Lean 4 formalization covers its algebraic core.

References

[1] Stefano Canino, Alex Casarotti, and Pierpaola Santarsiero. A new bound on the rank of tensor product of W-states, 2025. URL https://arxiv.org/abs/2512.05828. arXiv:2512.05828v1, Theorem 1.1.

[2] Open Problems in Numerical Linear Algebra. TR-29: Tensor computations, 2026. URL https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/tensor-computations/TR-29. Online problem catalogue.