The Entropy Law for the Asymptotic Subrank of Dicke Tensors

Sketch of proof

Wenqi Zhu[1], Chunlin Liu[2], Kai Zou[3] and Ping Nie[4]

[1]Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK. wenqi.zhu@maths.ox.ac.uk
[2]School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, China
[3]Mathematical & Engineering Sciences, King's College London, UK. kz@netmind.ai
[4]David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada.
ping.nie@uwaterloo.ca

The question and our answer. Fix $n \geq 2$ and positive integers $\mathbf{d} = (d_1, \ldots, d_n)$; put $k = \sum_a d_a$ and $p_a = d_a/k$. The *Dicke tensor* $D_{\mathbf{d}} \in (\mathbb{C}^n)^{\otimes k}$ is the sum of all basis words $e_{a_1} \otimes \cdots \otimes e_{a_k}$ in which symbol a appears exactly d_a times. Oneto and Ventura [1, Conj. 4.26] (see also [2, 3] and catalogue problem TR-28 [4]) conjectured that its asymptotic subrank is the entropy of p. We prove, for every $\mathbf{d}$, with all k modes kept separate,

$$\boxed{\widetilde{Q}_{\mathrm{M}}(D_{\mathbf{d}}) = \widetilde{Q}(D_{\mathbf{d}}) = \exp \mathrm{H}(p) = 2^{\mathrm{H}_2(p)}, \qquad Q(D_{\mathbf{d}}^{\boxtimes N}) \leq k\, e^{N\,\mathrm{H}(p)} \quad (N \geq 1).}$$

Here Q allows arbitrary linear maps, while Q_{M} allows only deleting, relabelling and rescaling coordinates. The lower bound (Steps 2–5) is built with coordinate restrictions; the upper bound (Step 6) holds for arbitrary linear maps. The two meet.

1. Powers, limits and induced matchings

A coordinate restriction of $D_{\mathbf{d}}^{\boxtimes N}$ that leaves q support entries with distinct coordinates in every mode, and no other entry, is a diagonal of size q; we call such a set an *induced matching*. Tensoring restrictions gives supermultiplicativity

$$R(D_{\mathbf{d}}^{\boxtimes (N+L)}) \;\geq\; R(D_{\mathbf{d}}^{\boxtimes N})\, R(D_{\mathbf{d}}^{\boxtimes L}), \qquad R \in \{Q, Q_{\mathrm{M}}\},$$

and $1 \leq R \leq n^N$. By Fekete's lemma the limits exist and equal $\sup_N R^{1/N}$, so **a lower bound along any unbounded subsequence of N suffices**. The case $k = 2$ is a 2×2 permutation matrix, so we assume $k \geq 3$.

2. An entropy inequality from symmetry

For a finite random vector X in a rational space E and a subspace V, let $F_X(V) = H(X) - H(X \bmod V)$ be the entropy that V "hides". F_X is *supermodular*: $F_X(U) + F_X(V) \leq F_X(U + V) + F_X(U \cap V)$. If a group acts irreducibly on E (with $\dim E = t$) and the law of X is invariant, a maximal-ratio argument shows

$$\boxed{F_X(V) \;\leq\; \frac{\dim V}{t}\, H(X) \qquad \text{for every subspace } V.} \tag{1}$$

In words: no subspace can hide more than its fair share of entropy.

3. Counting collisions

Label symbol a by the distinct integer $\alpha_a = ka - \sum_b bd_b$, so every support column $s \in S$ (a k-tuple with multiplicities $\mathbf{d}$) has coordinate sum zero. Let Z be uniform on S, so $H(Z) = h := \log M$ with $M = k!/\prod d_a!$. For a mode j let $E_j = \{v : \sum_i v_i = 0,\ v_j = 0\}$ and

$$\beta = \frac{h - \mathrm{H}(p)}{k-2}.$$

Conditioning on Z_j and applying (1) to the permutation-invariant conditional laws (the standard action of S_{k-1} on E_j is irreducible) gives

$$\boxed{F_Z(V) \le (\dim V)\,\beta \qquad \text{for } V \subseteq E_j.} \tag{2}$$

Candidates are the support sequences $I \in S^N$ of full joint type (each $s \in S$ used N/M times); there are $|A_N| \ge M^N/(N+1)^M$ of them. For a fixed candidate, the number of *all* original edges J with $J(t) - I(t) \in V$ for every t is exactly $\exp(NF_Z(V)) \le \exp(N \dim V\, \beta)$. Partners are counted in the whole support, not only among candidates.

4. Rank-dependent hashing

Take a large prime q and an *average-free* set $B \subseteq \{0, \dots, \lfloor (q-1)/(k-1) \rfloor\}$, i.e. $x_1 + \cdots + x_{k-1} = (k-1)y$ in B forces all equal, with $\log|B| = \log q - O(\sqrt{\log q})$ (a Behrend-type sphere construction). Hash local words by random affine functions over $\mathbb{F}_q$,

$$b_i(w) = u_i + \textstyle\sum_t w_t v_t \ (i < k), \qquad b_k(w) = \big(\sum_{i<k} u_i - \sum_t w_t v_t\big)/(k-1),$$

and keep only words whose hash lies in B. Because columns sum to zero, a surviving edge has all k hashes equal. One computes exactly

$$\boxed{\Pr[I \text{ survives}] = |B|\, q^{-(k-1)}, \qquad \Pr[I, J \text{ survive}] = |B|\, q^{-(k-1+r)},} \tag{3}$$

where $r = \dim \mathrm{span}\{I(t) - J(t)\}$, as long as q exceeds a fixed bound on the minors (so ranks over $\mathbb{Q}$ and $\mathbb{F}_q$ agree).

5. Isolation and the lower bound

Let X count surviving candidates and Y count pairs (surviving candidate, other surviving edge sharing a word with it). Summing Step 3 against (3) over the finitely many possible difference spans,

$$\mathbb{E}\,Y \;\le\; \mathbb{E}\,X \sum_{V} \exp(N \dim V\, \beta)\, q^{-\dim V}.$$

Keeping only the isolated candidates is a genuine coordinate restriction whose support is exactly those candidates, so $Q_{\mathrm{M}}(D_{\mathbf{d}}^{\boxtimes N}) \ge \mathbb{E}(X - Y)$. Fix $\delta > 0$, set $\lambda = \beta + \delta$, and choose $N \approx \log q/\lambda$ (a multiple of M). Then $\mathbb{E}Y \le Ke^{-N\delta}\,\mathbb{E}X$ and

$$\log Q_{\mathrm{M}}(D_{\mathbf{d}}^{\boxtimes N}) \;\ge\; Nh - (k-2)\log q - o(N) \;=\; N\big[\mathrm{H}(p) - (k-2)\delta\big] - o(N).$$

By Step 1 and $\delta \to 0$:

$$\boxed{\widetilde{Q}(D_{\mathbf{d}}) \;\ge\; \widetilde{Q}_{\mathrm{M}}(D_{\mathbf{d}}) \;\ge\; \exp \mathrm{H}(p).} \tag{4}$$

6. The matching upper bound by slice rank

Subrank is at most slice rank: $Q(T) \le \mathrm{sr}(T)$, since $\mathrm{sr}(\Delta_q) = q$ (proved via the span inequality $\dim(U * V) \ge \dim U + \dim V - q$) and linear maps send slices to slices. Give each word w of length N the score $s(w) = -\sum_t \log p_{w_t}$ and let $L_N = \{w : s(w) \le N\,\mathrm{H}(p)\}$; since $\sum_w \prod_t p_{w_t} = 1$, we get $|L_N| \le e^{N\,\mathrm{H}(p)}$. Every support edge has total score $\sum_i s(w^{(i)}) = Nk\,\mathrm{H}(p)$, so at least one of its k words lies in L_N. Grouping edges by that word gives at most $k|L_N|$ slices:

$$\boxed{Q(D_{\mathbf{d}}^{\boxtimes N}) \;\le\; \mathrm{sr}(D_{\mathbf{d}}^{\boxtimes N}) \;\le\; k\, e^{N\,\mathrm{H}(p)}.} \tag{5}$$

Taking N-th roots and combining with (4) proves $\widetilde{Q}_{\mathrm{M}}(D_{\mathbf{d}}) = \widetilde{Q}(D_{\mathbf{d}}) = \exp \mathrm{H}(p)$. For example, equal multiplicities give n, and $\mathbf{d} = (k-1, 1)$ gives $k/(k-1)^{(k-1)/k}$.

References

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[4] Open Problems in Numerical Linear Algebra, problem TR-28. https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/tensor-computations/TR-28.