Secant Varieties of Segre Products Need Not Be Cohen–Macaulay

Sketch of proof

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The conjecture and our answer. Let $X_r(a,b,c) \subseteq \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ be the reduced variety of tensors of border rank at most r, the affine cone over the secant variety $\sigma_r(\mathrm{Seg}(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}))$. Oeding conjectured that all secant varieties of Segre products are arithmetically Cohen–Macaulay [1]; this is catalogue problem TR-25 [3]. We prove that the conjecture fails in a whole family:

$$\begin{array}{c} r \geq 5, \quad a,b \geq 3, \quad a+b \geq r+2, \quad c \geq r \\ \Longrightarrow \; X_r(a,b,c) \text{ fails } S_2, \text{ and } \operatorname{depth} R_{\mathfrak{m}} < D = r(a+b+c-2). \end{array}$$

This includes $X_5(3,4,5)$, $X_5(6,6,6)$ and every unit-tensor orbit closure $X_r(r,r,r)$, $r \geq 5$. The mechanism in one sentence: one tensor with *two* different decompositions, both immersive, splits the normalization into two branches, and Serre's condition S_2 cannot glue them along a set of codimension two.

1. The decomposition space and the addition map

Let $\mathcal{P}$ be the smooth cone of nonzero pure tensors, of dimension $a+b+c-2$. Take unordered r-tuples of pure tensors whose third factors are linearly independent; this is a smooth irreducible variety Y of dimension $D = r(a+b+c-2)$. Addition gives

$$f : Y \longrightarrow X_r, \qquad \{p_1, \ldots, p_r\} \mapsto p_1 + \cdots + p_r. \tag{1}$$

We will show that f is birational, that it maps a large open set Y_{good} into the smooth locus, and that $\operatorname{codim}_Y(Y \setminus Y_{\mathrm{good}}) \geq 2$.

2. A rank condition that forces smoothness

Write the i-th summand as $a_i \otimes b_i \otimes c_i$ and put $P_i = a_i \otimes b_i$. Require, for every subset J with $|J| \geq 2$,

$$\operatorname{rank}(a_i : i \in J) + \operatorname{rank}(b_i : i \in J) \;\geq\; |J| + 2. \tag{G}$$

By Sylvester's rank inequality, (G) implies that $L = \operatorname{span}(P_i)$ contains exactly the r rank-one directions $[P_1], \ldots, [P_r]$, each with $L \cap T_{P_i} = \mathbb{C}P_i$; so $\mathbb{P}(L)$ meets the Segre variety in r reduced points. A finite étale incidence argument then shows the span variety is smooth at L, hence X_r is smooth at the tensor, and the decomposition there is unique. A determinantal count shows that failure of (G) has codimension at least two when $a,b \geq 3$ and $a+b \geq r+2$. Thus f is birational, $\dim X_r = D$, and

$$f(Y_{\mathrm{good}}) \subseteq X_r^{\mathrm{sm}}, \qquad \operatorname{codim}_Y(Y \setminus Y_{\mathrm{good}}) \geq 2. \tag{2}$$

3. The conductor lemma

Let A be a local domain essentially of finite type over $\mathbb{C}$, let B be its finite normalization, and let $I = (A : B)$ be the conductor.

$$\boxed{A \text{ satisfies } S_2 \implies \text{every point of } V(I) \subseteq \operatorname{Spec} B \text{ lies on a height-one component.}} \tag{3}$$

The reason: $I \cong \operatorname{Hom}_A(B, A)$ inherits S_2 as a B-module, and if a minimal prime $\mathfrak{h}$ of I had height ≥ 2, the depth lemma applied to $0 \to I_{\mathfrak{h}} \to B_{\mathfrak{h}} \to (B/I)_{\mathfrak{h}} \to 0$ would give a zero-dimensional module of positive depth. Also, if the normalization has two points over the closed point, then I is proper there.

4. Two immersive decompositions obstruct S_2

Suppose a tensor T has two distinct decompositions $y_1 \neq y_2$ in Y, and df is injective at both. Lift f to $g : Y \to \overline{X}$, the normalization. Injectivity makes g quasi-finite near y_1, y_2, and Zariski's Main Theorem identifies a neighbourhood of each with an open chart of $\overline{X}$: **two distinct normalization points above** T. If $\mathcal{O}_{X_r,T}$ were S_2, Step 3 would put a conductor divisor through these points. But on the chart the conductor is supported outside Y_{good}, i.e. in codimension ≥ 2 by (2). Contradiction:

$$\boxed{\mathcal{O}_{X_r,T} \text{ is not } S_2, \text{ hence not Cohen–Macaulay.}} \tag{4}$$

5. The explicit witness

Everything now rests on one concrete tensor. Let

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}, \qquad Q_i = e_i \operatorname{row}_i(M), \quad Q_{3+j} = -\operatorname{col}_j(M)\, e_j^{\mathsf{T}} \quad (i,j = 1,2,3).$$

These six rank-one matrices satisfy $Q_1 + \cdots + Q_6 = 0$ and no other relation, so any five span the same 5-dimensional space L_0. Six nonzero rational 3×3 determinants ($-\frac{1}{18}, \frac{1}{3}, -2$ for rows; $-\frac{1}{18}, \frac{4}{9}, -\frac{3}{2}$ for columns) show $L_0 \cap T_{Q_s} = \mathbb{C}Q_s$ for all six. Now set

$$\boxed{T_5 = \sum_{i=1}^{5} Q_i \otimes f_i = \sum_{i=1}^{4} Q_i \otimes (f_i - f_5) - Q_6 \otimes f_5.} \tag{5}$$

These are two different five-term decompositions. The tangent condition makes both immersive (differential rank 45 in $3\times3\times5$; after padding, 50 in $3\times4\times5$ and 80 in $6\times6\times6$), and the third flattening has rank 5, so rank and border rank are both 5.

For $r \geq 6$, add $r-5$ rank-one matrices $u_s \otimes v(t_s)$ and $v(h_t) \otimes w_t$ in new rows and columns, with $v(t) = (1, t, t^2)$ and distinct integers $t \geq 4$, each paired with a new third basis vector. Two determinant checks show these new directions do not meet the old tangent spaces, so both decompositions stay immersive; padding reaches every format in the theorem.

6. From the witness to the vertex, and the limits

If $R = \mathbb{C}[X_r]$ were Cohen–Macaulay at the homogeneous maximal ideal $\mathfrak{m}$, a minimal graded free resolution and Auslander–Buchsbaum would make every localization Cohen–Macaulay, including the one at T. Hence

$$\boxed{\operatorname{depth} R_{\mathfrak{m}} \;<\; D = \dim R_{\mathfrak{m}},} \tag{6}$$

and the projective secant is not locally Cohen–Macaulay at $[T]$, so it is not arithmetically Cohen–Macaulay. This disproves Oeding's conjecture in the stated family.

What the method does not say. Rank five is the first rank at which two immersive decompositions can occur in three factors, so it is sharp *for this mechanism*; it does not classify fourth secants. The neighbouring family $X_5(3,3,c)$ is Cohen–Macaulay for every c (it is the ambient space or a generic determinantal variety). The cases $X_5(3,4,4)$ and $X_5(4,4,4)$ are not decided. Bürgisser and Ikenmeyer had shown that the unit-tensor orbit closures are not normal [2]; our result upgrades this to failure of S_2.

References

[1] L. Oeding. *Are all secant varieties of Segre products arithmetically Cohen–Macaulay?* arXiv:1603.08980v2, Conjecture 1.1.

[2] P. Bürgisser, C. Ikenmeyer. *Geometric complexity theory and tensor rank.* STOC 2011, 509–518, Corollary 7.2.

[3] Open Problems in Numerical Linear Algebra, problem TR-25. https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/tensor-computations/TR-25.