Secant Varieties of Segre Products Need Not Be Cohen–Macaulay

Two immersive decompositions, the conductor, and a failure of Serre's condition S_2

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Abstract

For every $r \geq 5$, $a, b \geq 3$, $a + b \geq r + 2$, and $c \geq r$, the reduced affine variety of tensors of border rank at most r in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ fails Serre's condition S_2 and is not Cohen–Macaulay. Its dimension is $D = r(a+b+c-2)$ and its homogeneous vertex has depth less than D. This includes the rank-five variety in format $3 \times 4 \times 5$ and the closure of the unit-tensor orbit in format $r \times r \times r$ for every $r \geq 5$. The corresponding projective secants are locally non-Cohen–Macaulay, disproving the universal arithmetic Cohen–Macaulay conjecture for Segre secants. Explicit integer tensors have two immersive decompositions. A rank condition on matrix factors yields smooth target charts with a codimension-two complement in a birational decomposition space; the conductor of the finite normalization then obstructs S_2. We prove that rank five is the first rank admitting this three-factor decomposition mechanism. We also identify the adjacent Cohen–Macaulay family in format $3 \times 3 \times c$ at rank five and prove that the mechanism cannot occur on an invertible-slice chart. Exact arithmetic and a Lean formalization certify the fixed finite witness calculations; the infinite family follows from the written proof.

1 The main theorem and its context

For positive integers a, b, c, r, let

$$X_r(a,b,c) = \overline{\left\{ \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i : a_i \in \mathbb{C}^a,\ b_i \in \mathbb{C}^b,\ c_i \in \mathbb{C}^c \right\}}^{\mathrm{Zar}}, \tag{1.1}$$

with its reduced structure. It is the affine cone over $\sigma_r(\mathrm{Seg}(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}))$. The coordinate ring is a standard graded domain, since the parametrizing space is irreducible and its image closure is conical. All varieties and tensor products in the proofs are over $\mathbb{C}$.

Theorem 1.1. *Suppose that*

$$r \geq 5, \qquad a, b \geq 3, \qquad a + b \geq r + 2, \qquad c \geq r. \tag{1.2}$$

Set $X = X_r(a,b,c)$, $R = \mathbb{C}[X]$, and $\mathfrak{m} = R_{>0}$. Then

$$\dim X = D = r(a+b+c-2),$$

R does not satisfy Serre's condition S_2, and R is not Cohen–Macaulay. Its homogeneous local ring satisfies

$$\operatorname{depth} R_{\mathfrak{m}} < D = \dim R_{\mathfrak{m}}. \tag{1.3}$$

There is an explicit integer tensor $T \in X$, given by (4.18) and (4.20), of ordinary rank and border rank exactly r for which $\mathcal{O}_{X,T}$ is not Cohen–Macaulay. The same assertions hold after permuting the three factor dimensions.

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Corollary 1.2. *The varieties $X_5(3,4,5)$ and $X_5(6,6,6)$ fail S_2, have respective dimensions 50 and 80, and have homogeneous vertex depths strictly smaller than those dimensions. For every $r \geq 5$, $X_r(3, r-1, r)$ has dimension $2r^2$ and fails S_2. Moreover, if $\Delta_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$, then*

$$X_r(r,r,r) = \overline{\left(\mathrm{GL}_r(\mathbb{C})\right)^3 \cdot \Delta_r} \tag{1.4}$$

fails S_2 and has vertex depth less than $3r^2 - 2r$.

Corollary 1.3. *For every format in (1.2), the projective secant $\sigma_r(\mathrm{Seg}(\mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}))$ has a non-Cohen–Macaulay local ring at the class of the tensor in Theorem 1.1. In particular, it is not arithmetically Cohen–Macaulay.*

A Noetherian ring B satisfies S_2 when

$$\operatorname{depth} B_{\mathfrak{p}} \geq \min\{2, \dim B_{\mathfrak{p}}\} \quad \text{for every } \mathfrak{p} \in \operatorname{Spec} B.$$

Cohen–Macaulay rings satisfy S_2. Arithmetic Cohen–Macaulayness of a projective embedding means Cohen–Macaulayness of its homogeneous coordinate ring, including the affine vertex. It is stronger than local Cohen–Macaulayness of the projective variety; here both fail. Theorem 1.1 does not specify the exact depth at the witness or at the vertex. Failure of S_2 for $\mathcal{O}_{X,T}$ may occur at a further localization and is not an assertion that $\operatorname{depth} \mathcal{O}_{X,T} < 2$.

1.1 The conjecture and established cases

Landsberg and Weyman asked whether secant varieties of Segre varieties are arithmetically Cohen–Macaulay [6, Introduction]. Oeding formulated the universal assertion as Conjecture 1.1 of [7]. Jagiełła and Jelisiejew restated it as Conjecture 3.32 of [5], explicitly identifying arithmetic Cohen–Macaulayness with Cohen–Macaulayness of the affine cone. Both formulations concern the ordinary reduced secant variety of a Segre product of projective spaces. They impose neither a conciseness condition nor a restriction to balanced formats. Theorem 1.1 refutes this assertion with the same field, reduced structure, and projective embedding.

Several substantial positive families are known. Landsberg and Weyman proved arithmetic Cohen–Macaulayness for all secants of $\mathbb{P}^1 \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}$ with $b, c \geq 2$, second secants of four Segre factors of vector dimensions at least two, and third secants of three Segre factors of vector dimensions at least three [6, Theorems 1.1–1.3]. For the first family their proof gives the affine cone rational singularities, and hence the arithmetic conclusion. Catalisano, Geramita and Gimigliano proved further unbalanced cases by identifying the secants with generic determinantal varieties [3, Theorem 2.4]. Oeding proved that $\sigma_4(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^n))$ is arithmetically Cohen–Macaulay for every $n \geq 3$ [7, Theorem 3.2]. For arbitrary numbers of Segre factors, Raicu proved that the ideal of the secant-line variety is generated by the cubic flattening minors [8, Corollary 4.2]; this is an ideal-generation result, not an arithmetic Cohen–Macaulayness assertion. These results describe selected positive cases and known equations, not a classification of the Cohen–Macaulay secants.

1.2 Comparison with nonnormality

For $r \geq 1$, let $\Delta_r = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$. Bürgisser and Ikenmeyer proved that the unit-tensor orbit closure

$$\overline{\left(\mathrm{GL}_r(\mathbb{C})\right)^3 \cdot \Delta_r} \subseteq (\mathbb{C}^r)^{\otimes 3}$$

is not normal for every $r \geq 5$ [1, Corollary 7.2(2)]; their later treatment repeats the result [2, Corollary 5.26]. The corollary numbers refer to the arXiv versions specified in the bibliography. The orbit closure equals $X_r(r,r,r)$: the invertible triples of factor matrices form a dense open subset of the parametrizing space for sums of r pure tensors. Theorem 1.1 therefore strengthens the known nonnormality of these same orbit closures to failure of S_2 and of Cohen–Macaulayness, for every $r \geq 5$.

Nonnormality alone does not imply either conclusion. For example, $\mathbb{C}[t^2, t^3] \simeq \mathbb{C}[u,v]/(v^2 - u^3)$ is a Cohen–Macaulay hypersurface domain that is not normal: t is integral over the ring but does not belong to it. Serre's normality criterion separates the conditions R_1 and S_2 [10, Tag 031S]. The proof below detects failure of S_2 through the conductor of the finite normalization; the cited nonnormality results are not assumptions in that proof.

1.3 The proof and the limits of the mechanism

The proof separates smoothness of suitable target points from the ambiguity at one explicit tensor. A condition on the ranks of all subsets of two factor lists forces their matrix span to contain exactly the expected reduced rank-one points. A proper incidence argument then reconstructs a smooth local configuration space. Keeping the third factor list independent gives a smooth birational decomposition source whose complement of smooth-image points has codimension at least two.

The integer tensor has two distinct decompositions where addition is immersive. They produce two distinct open charts in the finite normalization. If the tensor secant were S_2 at that tensor, its conductor would contain a divisor through the corresponding normalization point. The codimension estimate excludes such a divisor. Adding explicit matrices in new rows and columns extends the construction to every rank in (1.2).

The range in the theorem is sufficient, not a classification of all non-Cohen–Macaulay secants. We prove that two distinct immersive r-term decompositions cannot occur in three factors for $r \le 4$; this makes $r = 5$ sharp for this mechanism. It does not assert that all fourth secants are arithmetically Cohen–Macaulay. The rank-five family $X_5(3,3,c)$ remains Cohen–Macaulay for every $c \ge 1$. We also show why the displayed ambiguity cannot be transported through an invertible-slice chart to give a Hilbert- or Quot-scheme counterexample. No global minimal-format assertion, classification in higher tensor order, or Cohen–Macaulay assertion about the normalizations is made.

2 Matrix spans and the decomposition morphism

Throughout this section, let A, B, C be complex vector spaces of dimensions a, b, c, and suppose that

$$r \ge 5, \qquad a, b \ge 3, \qquad a + b \ge r + 2, \qquad c \ge r. \tag{2.1}$$

Write $W = A \otimes B$ and

$$S = \mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B) \subseteq \mathbb{P}W.$$

We regard W as the space of linear maps $B^* \to A$; its nonzero rank-one elements are the nonzero points of the affine cone $\widehat{S}$. We construct a smooth space of unordered decompositions which maps birationally to $X_r(a,b,c)$. The construction also supplies an open subset whose complement has codimension at least two and whose image consists of smooth points of $X_r(a,b,c)$.

2.1 Rank-one points in a matrix span

For nonzero vectors $a_i \in A$ and $b_i \in B$, put $P_i = a_i \otimes b_i$, $1 \le i \le r$. For a subset $J \subseteq \{1, \dots, r\}$, let A_J and B_J be the matrices with columns a_i and b_i, respectively, indexed by J. Consider the condition

$$\operatorname{rank} A_J + \operatorname{rank} B_J \ge |J| + 2 \qquad \text{whenever } |J| \ge 2. \tag{G}$$

This condition is invariant under rescaling any of the factor vectors and under permuting the indices.

Lemma 2.1. *Suppose that the P_i satisfy (G), and put $L = \langle P_1, \dots, P_r \rangle \subseteq W$. Then the P_i are linearly independent, and the only rank-one directions in L are $\mathbb{C}P_1, \dots, \mathbb{C}P_r$. Moreover, for every i,*

$$L \cap T_{P_i}\widehat{S} = \mathbb{C}P_i. \tag{2.2}$$

Consequently the scheme-theoretic intersection $\mathbb{P}L \cap S$ consists of the r distinct reduced points $[P_1], \dots, [P_r]$.

Proof. Take a linear combination with nonzero coefficients indexed by a nonempty subset J, and write $t = |J|$. In matrix notation it is

$$N = \sum_{i \in J} \lambda_i P_i = A_J \operatorname{diag}(\lambda_i : i \in J) B_J^{\mathsf{t}}.$$

Sylvester's rank inequality gives

$$\operatorname{rank} N \ge \operatorname{rank} A_J + \operatorname{rank} B_J - t. \tag{2.3}$$

Indeed, the image of the right-hand map has dimension $\operatorname{rank} B_J$, and its intersection with the kernel of the left-hand map has dimension at most $t - \operatorname{rank} A_J$. If $t \ge 2$, the right-hand side of (2.3) is at least two. If

$t = 1$, the combination is a nonzero rank-one matrix. No nonempty combination can therefore vanish, proving independence, and a rank-one combination can use only one of the P_i.

Fix i. The affine tangent space at P_i is

$$T_{P_i}\widehat{S} = a_i \otimes B + A \otimes b_i.$$

It is the kernel of the quotient map

$$W \longrightarrow (A/\mathbb{C}a_i) \otimes (B/\mathbb{C}b_i).$$

We show that the images of P_j, $j \neq i$, under this map are linearly independent. Let J be the nonempty support of a proposed relation among these images, and put $t = |J|$. The ranks of the two projected factor matrices are

$$\operatorname{rank} A_{J\cup\{i\}} - 1, \qquad \operatorname{rank} B_{J\cup\{i\}} - 1.$$

Their sum is at least $t + 1$ by (G). Applying the same rank inequality in the quotient spaces shows that the proposed combination has rank at least one. This proves independence and hence (2.2).

The set-theoretic intersection $\mathbb{P}L \cap S$ has already been identified. At each of its points $[P_i]$, its Zariski tangent space is the quotient of $L \cap T_{P_i}\widehat{S}$ by $\mathbb{C}P_i$, and is therefore zero. If R is the local ring of the intersection at this point and $\mathfrak{m}$ its maximal ideal, this says $\mathfrak{m}/\mathfrak{m}^2 = 0$. Nakayama's lemma gives $\mathfrak{m} = 0$, so $R = \mathbb{C}$. Thus all the intersection points are reduced, including their scheme structures. □

Lemma 2.2. *In*

$$\mathcal{F}_{A,B} = (A \setminus \{0\})^r \times (B \setminus \{0\})^r,$$

the locus where (G) *fails has codimension at least two. In particular, the condition holds on a nonempty open subset of* $\mathcal{F}_{A,B}$.

Proof. For each fixed subset J of size $t \geq 2$, a bad pair of ranks is contained in a locus

$$\operatorname{rank} A_J \leq x, \qquad \operatorname{rank} B_J \leq y, \qquad x + y \leq t + 1,$$

where $1 \leq x \leq \min(a, t)$ and $1 \leq y \leq \min(b, t)$. Zero is excluded because the individual columns are nonzero. The two determinantal conditions involve disjoint sets of coordinates, and their codimensions in the full matrix spaces add to

$$(a - x)(t - x) + (b - y)(t - y). \tag{2.4}$$

This uses the dimension $x(a+t-x)$ of the variety of a-by-t matrices of rank at most x. One obtains that dimension by choosing the x-dimensional column space and then a surjective map from $\mathbb{C}^t$ onto it; the smaller-rank strata have smaller dimension. Intersecting with the open conditions on the columns either removes a stratum or preserves its codimension.

For $2 \leq t \leq r$, the numerical assumptions imply

$$\min(a, t) + \min(b, t) \geq t + 2. \tag{2.5}$$

To see this, if both a, b are at least t, the sum is $2t$; if exactly one is smaller than t, the sum is at least $t + 3$; and if both are smaller, their sum is at least $r + 2 \geq t + 2$. Thus (2.4) cannot be zero for a bad pair.

Nor can it be one. If, for example, its first summand is one and the second is zero, then $a = t$, $x = t - 1$, and $y = \min(b, t)$. Since $a = t \geq 3$ and $b \geq 3$, we have $y \geq 3$ and hence $x + y \geq t + 2$, a contradiction. The other possibility is symmetric. Every bad determinantal locus therefore has codimension at least two. There are only finitely many choices of J, x, y, proving the asserted codimension bound. The bad locus is closed as a finite union of these closed determinantal loci; its complement is consequently a nonempty open subset. □

2.2 The variety of spans

Let $\mathcal{U} \subseteq S^r$ be the open subset of ordered tuples of linearly independent projective points. Here independence means that nonzero representatives in W are linearly independent; in particular the points are distinct. There is a span morphism

$$s : \mathcal{U} \longrightarrow \operatorname{Gr}(r, W), \qquad ([P_1], \ldots, [P_r]) \longmapsto \langle P_1, \ldots, P_r \rangle.$$

Let $H \subseteq \operatorname{Gr}(r, W)$ be the reduced closure of its image. The preceding two lemmas show that $\mathcal{U}$ is nonempty. It is smooth and irreducible of dimension $r(a + b - 2)$, and H is integral. We regard s as a dominant morphism to H.

Lemma 2.3. *Suppose that $L \in H$ and that the scheme $\mathbb{P}L \cap S$ consists of exactly r reduced points spanning $\mathbb{P}L$. Then H is smooth at L. More precisely, there is an open neighborhood N of L in H such that ordering the points of $\mathbb{P}K \cap S$, for $K \in N$, gives a finite* étale *surjective cover of N isomorphic to $s^{-1}(N) \subseteq \mathcal{U}$. In particular,*

$$\dim H = r(a+b-2). \tag{2.6}$$

Proof. Consider the scheme-theoretic incidence

$$Z = \{(K,p) \in H \times S : p \in \mathbb{P}K\}, \qquad \pi : Z \longrightarrow H.$$

It is closed in $H \times S$, so π is projective. Its fiber at L is the given reduced scheme of length r. Upper semicontinuity of fiber dimension for a proper morphism supplies an open neighborhood of L on which all fibers are zero-dimensional. On this neighborhood π is proper and quasi-finite, and hence finite.

We first verify that its generic fiber has length r. Condition (G) defines a nonempty open subset of $\mathcal{U}$: this follows by descending the invariant condition on nonzero factor vectors, using Lemma 2.2. Its image under s is dense and constructible in H, and therefore contains a dense open subset. At every closed point of this image, Lemma 2.1 identifies the incidence fiber as exactly r reduced points. On an affine open where π is finite, its algebra is a finite module over a domain. Such a module becomes free after inverting a nonzero element: choose a basis over the fraction field and clear denominators in generators and relations. The rank of this free module is the length of every fiber on that smaller open. Comparing at a point in the dense image just described shows that the generic length is r.

Set $R = \mathcal{O}_{H,L}$, and let F be the finite R-algebra obtained from Z by base change to $\operatorname{Spec} R$. Since

$$\dim_{\mathbb{C}} F/\mathfrak{m}_R F = r,$$

lifting a basis of this fiber and applying Nakayama's lemma gives a surjection $R^r \twoheadrightarrow F$. After tensoring with the fraction field of R, this is a surjection between vector spaces of dimension r, and hence an isomorphism. Its kernel has rank zero. But the kernel is a submodule of the free module R^r over the domain R, so it is torsion-free; consequently it is zero. We have proved

$$F \simeq R^r \qquad \text{as an } R\text{-module.} \tag{2.7}$$

After shrinking about L, the incidence morphism is therefore finite locally free of rank r.

The fiber at L is a disjoint union of r copies of $\operatorname{Spec}\mathbb{C}$, and is thus *étale* over $\mathbb{C}$. A flat morphism of finite presentation with an *étale* fiber is *étale* at the points of that fiber. The non-*étale* locus is closed, and its image under the finite morphism π is closed. Removing that image yields an open neighborhood N of L on which

$$Z_N \longrightarrow N$$

is finite *étale* of degree r.

Form the cover $Q \to N$ which orders its r points. Concretely, Q is the complement of the pairwise diagonals in the r-fold fiber product of Z_N over N. For a finite *étale* morphism the diagonals are both open and closed, so Q is finite *étale* over N; each geometric fiber has $r!$ points. The condition that the ordered points span the corresponding r-dimensional subspace is open on Q. It holds at every point above L. Since $Q \to N$ is finite, we can again remove the image of its closed complement and arrange that it holds throughout Q.

The ordered incidence points now define a morphism $Q \to \mathcal{U}$. Its image lies in $s^{-1}(N)$, and taking their span recovers the given point of N. Conversely, a tuple in $s^{-1}(N)$ supplies r distinct points of the corresponding incidence fiber. That fiber is finite *étale* of degree r, so the tuple orders all of it and gives a point of Q. These constructions are inverse morphisms. Indeed, on families the independence condition identifies the direct sum of the r tautological lines with the pulled-back rank-r tautological bundle, so recovering the span is an identity of subbundles, not merely an equality at closed points. Thus

$$Q \simeq s^{-1}(N). \tag{2.8}$$

The right-hand side is a nonempty open subset of the smooth variety $\mathcal{U}$. Smoothness descends through the surjective *étale* morphism $Q \to N$, proving the smoothness assertion. The same finite *étale* cover gives $\dim N = \dim \mathcal{U} = r(a+b-2)$ and hence (2.6). □

2.3 The full third flattening

Identify a tensor in $A \otimes B \otimes C$ with its third flattening $\phi : C^* \to W$. Let

$$\mathcal{M}_r = \{\phi \in \mathrm{Hom}(C^*, W) : \mathrm{rank}\, \phi = r\}.$$

Sending a map to its image defines a morphism

$$\rho : \mathcal{M}_r \longrightarrow \mathrm{Gr}(r, W).$$

If $\mathcal{E}$ is the tautological rank-r bundle on the Grassmannian, then $\mathcal{M}_r$ is the open subspace of the total space of $\mathrm{Hom}(C^*, \mathcal{E})$ consisting of surjective maps. Its fiber over L is the smooth open subset of $\mathrm{Hom}(C^*, L)$ of rank-r maps, of dimension rc.

Lemma 2.4. *The open locus of third flattening rank r in $X_r(a,b,c)$ is, with its reduced scheme structure,*

$$X_r(a,b,c)^{(r)} = \rho^{-1}(H). \tag{2.9}$$

If T belongs to this locus and its flattening image L has scheme-theoretic intersection $\mathbb{P}L \cap S$ consisting of exactly r reduced spanning points, then $X_r(a,b,c)$ is smooth at T. Moreover,

$$\dim X_r(a,b,c) = r(a+b+c-2). \tag{2.10}$$

Proof. Every sum of r pure tensors has third flattening rank at most r, and the same holds on its closure. Hence the rank-r locus is open in $X_r(a,b,c)$, and is closed as a subvariety of $\mathcal{M}_r$. It is nonempty: choose a tuple in $\mathcal{U}$ and pair its representatives with r independent vectors in C. If such a tensor is an actual sum $\sum_i P_i \otimes c_i$, its rank forces both the P_i and the c_i to be linearly independent. Its flattening image is therefore the span of r independent points of S. Actual sums are dense in $X_r(a,b,c)$; intersecting with its nonempty rank-r open subset preserves density. Since $\rho^{-1}(H)$ is closed in $\mathcal{M}_r$, this proves one inclusion in (2.9).

Conversely, let $L = s(u)$ for $u \in \mathcal{U}$, and choose rank-one representatives $P_1, \ldots, P_r$ of the tuple u. Every surjection $C^* \to L$ has a unique expression

$$\lambda \longmapsto \sum_{i=1}^{r} \lambda(c_i) P_i$$

with independent vectors $c_i \in C$, so it is an actual sum of r pure tensors. Thus $\rho^{-1}(s(\mathcal{U}))$ is contained in $X_r(a,b,c)^{(r)}$. The projection $\rho^{-1}(H) \to H$ is smooth and surjective, and therefore open. The inverse image of the dense subset $s(\mathcal{U})$ is consequently dense in $\rho^{-1}(H)$. Taking closures in $\mathcal{M}_r$ gives the reverse inclusion. The variety $\rho^{-1}(H)$ is reduced: locally it is an open subset of a vector bundle over the reduced variety H. Equality of the reduced closed subvarieties follows.

By Lemma 2.3, H is smooth at an image L of the stated kind. The bundle description makes $X_r(a,b,c)^{(r)}$ smooth at every tensor above L. As this is open in $X_r(a,b,c)$, the latter is smooth at T as well. Finally, $X_r(a,b,c)$ is irreducible and its rank-r locus is nonempty. Its dimension is therefore

$$\dim H + rc = r(a+b-2) + rc = r(a+b+c-2),$$

by (2.6). □

2.4 The open space of decompositions

Let $\mathcal{P} \subseteq A \otimes B \otimes C$ be the variety of nonzero pure tensors. It is the complement of the zero section in the tautological line bundle over $\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C$. Thus $\mathcal{P}$ is smooth and irreducible of dimension $a+b+c-2$.

Define an ordered decomposition space by

$$\widetilde{Y} = \left\{ (p_1, \ldots, p_r) \in \mathcal{P}^r : \begin{array}{l} p_i = a_i \otimes b_i \otimes c_i, \\ c_1, \ldots, c_r \text{ are linearly independent} \end{array} \right\}. \tag{2.11}$$

Independence is unaffected by the choices of factor vectors, so this defines a nonempty open subset of $\mathcal{P}^r$. It implies that the summands are pairwise distinct. Consequently $\mathfrak{S}_r$ acts freely on $\widetilde{Y}$, and its quotient

$$Y = \widetilde{Y}/\mathfrak{S}_r \tag{2.12}$$

exists as a variety; the quotient map is finite *étale*. For example, the free finite-group quotient theorem [10, Tag 07S7] applies to the quasi-projective variety $\widetilde{Y}$; its torsor is finite *étale* because the constant group scheme $\mathfrak{S}_r$ is so over $\mathbb{C}$. Addition induces a morphism

$$f : Y \longrightarrow X_r(a,b,c), \qquad \{p_1,\dots,p_r\} \longmapsto \sum_{i=1}^{r} p_i. \tag{2.13}$$

Let Y_{good} be the open subset on which the first two lists of factor vectors satisfy (G).

Proposition 2.5. *Under (2.1), the variety Y is smooth and irreducible of dimension*

$$D = r(a+b+c-2) = \dim X_r(a,b,c),$$

and f is birational. The open subset Y_{good} satisfies

$$\operatorname{codim}_Y(Y \setminus Y_{\mathrm{good}}) \geq 2, \qquad f(Y_{\mathrm{good}}) \subseteq X_r(a,b,c)_{\mathrm{sm}}. \tag{2.14}$$

Every tensor in $f(Y_{\mathrm{good}})$ has a unique unordered decomposition into r nonzero pure tensors.

Proof. Smoothness and dimension descend through the finite *étale* quotient from the nonempty open subset $\widetilde{Y}$ of $\mathcal{P}^r$. Irreducibility follows from that of $\widetilde{Y}$. The full addition morphism on $\mathcal{P}^r$ is dominant onto $X_r(a,b,c)$; restricting it to the dense open subset $\widetilde{Y}$ preserves dominance. Thus f is dominant.

For the codimension assertion, use factor-vector coordinates upstairs. The morphism

$$\big((A\setminus\{0\})\times(B\setminus\{0\})\times(C\setminus\{0\})\big)^r \longrightarrow \mathcal{P}^r$$

is a smooth surjective torus torsor of relative dimension $2r$. For each summand the scalar changes have the form $(\lambda,\mu,(\lambda\mu)^{-1})$. The inverse image of $\widetilde{Y}$ is obtained by imposing only that the third-factor matrix have rank r. In this space, failure of (G) still has codimension at least two by Lemma 2.2, since the condition involves only the A- and B-coordinates. The bad locus is invariant under the torus and the permutations. A smooth surjection of constant relative dimension preserves its codimension, as does a finite *étale* surjection. This proves the first assertion of (2.14).

Let $y \in Y_{\mathrm{good}}$ and write its sum as $T = \sum_i P_i \otimes c_i$. The P_i are independent by Lemma 2.1, and the c_i are independent by the definition of Y. Thus T has third flattening rank r with image $L = \langle P_1,\dots,P_r\rangle$. The same lemma identifies $\mathbb{P}L \cap S$ as r reduced spanning points. Lemma 2.4 proves that T is smooth in $X_r(a,b,c)$, establishing the second assertion of (2.14).

Suppose that T has another r-term decomposition. Its third flattening rank forces its third vectors and matrix factors to be independent, and forces those matrix factors to span the same space L. Each is rank one, so their projective directions must be exactly the r points $[P_i]$. After permuting the summands, write the new matrix factors as $\lambda_i P_i$. Comparing the coefficients in the basis (P_i) shows that the corresponding new third factors are $\lambda_i^{-1}c_i$. The full pure summands are therefore unchanged, proving uniqueness.

The dimension of the target is D by Lemma 2.4. The image of the dense open subset Y_{good} is dense and constructible, and hence contains a dense open subset of $X_r(a,b,c)$. The fibers there consist of single points by the uniqueness just proved. A dominant generically finite morphism of irreducible complex varieties induces a separable extension of function fields; its degree is the number of points in a general fiber. That degree is one here, so f is birational. □

The restriction to independent third-factor vectors is part of the definition of Y, including when $c = r$. Thus no codimension claim about the third-rank-deficient boundary of a larger decomposition space is needed in Proposition 2.5.

3 The conductor obstruction

We first establish the local algebra needed to distinguish failure of S_2 from nonnormality. Normalization is finite for domains essentially of finite type over $\mathbb{C}$ [10, Tags 035B and 035S]. If A is such a domain and B is its normalization in $K = \operatorname{Frac}(A)$, its conductor is

$$I = (A : B) = \{a \in A : aB \subseteq A\}. \tag{3.1}$$

It is a nonzero ideal of both A and B. Indeed, clearing denominators of finitely many A-module generators of B gives a nonzero element of I. The equality $I = \operatorname{Ann}_A(B/A)$ identifies $V(I) = \operatorname{Supp}_A(B/A)$ with the locus where normalization is not an isomorphism. We call this closed set the conductor locus.

Lemma 3.1 (A conductor divisor under S_2)**.** *Let A be a local domain essentially of finite type over $\mathbb{C}$, and assume that A satisfies S_2. Let B be its finite normalization and $I = (A : B)$. If $\mathfrak{q} \in \operatorname{Spec} B$ satisfies $IB_{\mathfrak{q}} \neq B_{\mathfrak{q}}$, then there is a height-one prime $\mathfrak{h} \subseteq \mathfrak{q}$ of B containing I.*

Proof. As a B-module,

$$I \simeq \operatorname{Hom}_A(B, A). \tag{3.2}$$

For completeness, tensoring an A-linear map $B \to A$ with K gives a K-linear endomorphism of K, hence multiplication by some $u \in K$. Evaluating at 1 shows $u \in A$, and the requirement $uB \subseteq A$ gives $u \in I$. Conversely multiplication by every $u \in I$ defines such a map. The natural B-action on the Hom module corresponds to the ideal structure in (3.2).

We show that I is an S_2 module over B. Fix $\mathfrak{q} \in \operatorname{Spec} B$ and put $\mathfrak{p} = \mathfrak{q} \cap A$. The heights of these primes agree. To check this in the present setting, write the extension as a localization of the normalization of an affine domain. The two affine domains have the same fraction field, and the residue fields at corresponding primes form a finite algebraic extension. The formula

$$\operatorname{ht} \mathfrak{p} = \operatorname{trdeg}_{\mathbb{C}} K - \operatorname{trdeg}_{\mathbb{C}} \kappa(\mathfrak{p})$$

and its counterpart in the normalization therefore give equal heights [10, Tags 00OT, 00P0 and 00P1]. The localizations involved do not remove any prime below the primes being considered.

If this common height is at least two, S_2 supplies an $A_{\mathfrak{p}}$-regular sequence $x, y \in \mathfrak{p}A_{\mathfrak{p}}$. Write $B_{\mathfrak{p}}$ for localization as an A-algebra. Finite presentation gives

$$I_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(B_{\mathfrak{p}}, A_{\mathfrak{p}}).$$

Multiplication by x is injective on this module. Suppose that $y\varphi = x\psi$ for two elements of it. For every $b \in B_{\mathfrak{p}}$ one has $y\varphi(b) \in xA_{\mathfrak{p}}$. Regularity of y modulo x implies $\varphi(b) \in xA_{\mathfrak{p}}$. Division by x defines an $A_{\mathfrak{p}}$-linear map $\eta : B_{\mathfrak{p}} \to A_{\mathfrak{p}}$, with $\varphi = x\eta$. Hence y is a nonzerodivisor modulo x on $I_{\mathfrak{p}}$. Localizing further at $\mathfrak{q}$ preserves this regular sequence; both elements remain in the maximal ideal there. In height one, torsion-freeness of the nonzero ideal I gives depth at least one. This proves the asserted S_2 property of I.

Choose a prime $\mathfrak{h}$ minimal over I and contained in $\mathfrak{q}$. Its height is positive because I is a nonzero ideal in a domain. If $\operatorname{ht} \mathfrak{h} \geq 2$, normality of B and the preceding paragraph imply

$$\operatorname{depth} B_{\mathfrak{h}} \geq 2, \qquad \operatorname{depth} I_{\mathfrak{h}} \geq 2.$$

Apply the depth lemma [10, Tag 00LX] to

$$0 \longrightarrow I_{\mathfrak{h}} \longrightarrow B_{\mathfrak{h}} \longrightarrow (B/I)_{\mathfrak{h}} \longrightarrow 0.$$

It gives $\operatorname{depth}(B/I)_{\mathfrak{h}} \geq 1$. Minimality of $\mathfrak{h}$ over I, however, makes $(B/I)_{\mathfrak{h}}$ a nonzero zero-dimensional local module. Such a module has depth zero. This contradiction proves $\operatorname{ht} \mathfrak{h} = 1$. □

Lemma 3.2. *Under the notation of (3.1), suppose A is local with maximal ideal $\mathfrak{n}$ and that $\operatorname{Spec} B$ has at least two points above $\mathfrak{n}$. Then the conductor is proper at every point above $\mathfrak{n}$.*

Proof. If $IB_{\mathfrak{q}} = B_{\mathfrak{q}}$ at a prime above $\mathfrak{n}$, some $c \in I$ avoids $\mathfrak{q}$. Since $c \in A$ and $\mathfrak{q} \cap A = \mathfrak{n}$, it is a unit in A. Thus $I = A$, so $B = A$, contrary to the existence of the two points. □

Proposition 3.3 (The two-point obstruction)**.** *Let X be an irreducible affine complex variety and let $f : Y \to X$ be a dominant birational morphism, with Y smooth and irreducible. Suppose there is an open subset $G \subseteq Y$ such that*

$$f(G) \subseteq X_{\mathrm{sm}}, \qquad \operatorname{codim}_Y(Y \setminus G) \geq 2.$$

If distinct closed points $y_1, y_2 \in Y$ have the same image $T \in X$ and both differentials df_{y_i} are injective, then $\mathcal{O}_{X,T}$ does not satisfy S_2 as a ring and is not Cohen–Macaulay. In particular X does not satisfy S_2.

Proof. Let $\nu : \overline{X} \to X$ be the finite normalization. The dominant morphism f lifts to a morphism

$$g : Y \longrightarrow \overline{X}, \qquad \nu \circ g = f.$$

Here Y is normal because it is smooth. The normalization lifting property is [10, Tag 035Q(4)]; in this setting the property follows directly from integral closure: pullbacks of integral rational functions are integral over the relevant local rings of Y and hence regular. Since f and ν are birational, g is birational as well.

Injectivity of df_{y_i} makes the Zariski tangent space of the fiber $f^{-1}(T)$ zero at y_i. The local ring of that fiber therefore has dimension zero, so f is quasi-finite at y_i. The fiber of g through y_i is contained in that fiber, and g is also quasi-finite there. Let $Y_0 \subseteq Y$ be the open quasi-finite locus of g, containing both y_i [10, Tag 01TI]. The morphism $g|_{Y_0}$ is separated, quasi-finite and birational with normal target. The corresponding form of Zariski's Main Theorem implies that it is an open immersion [10, Tags 05K0 and 0AB1]. In particular,

$$q_i = g(y_i) \in \overline{X}$$

are distinct points lying over T, and an open neighborhood of each q_i is identified with the corresponding open subset of Y.

Assume for contradiction that $A = \mathcal{O}_{X,T}$ satisfies S_2. Its normalization B is the localization of the affine normalization above T, and the points q_i give two distinct primes of B above the maximal ideal of A. The conductor also commutes with this localization: use (3.2) and the fact that the finite normalization is a finitely presented module over the Noetherian coordinate ring. Lemmas 3.1 and 3.2 give a height-one prime in the conductor locus whose closure passes through q_1.

The same height-one prime defines a divisor on a neighborhood of q_1 in the affine normalization. Indeed, localization from the affine normalization to B preserves heights for all primes below q_1. The open chart containing q_1 also contains that height-one generic point. Consequently the chart $g(Y_0)$ contains a conductor divisor.

On the other hand, $f(G)$ is contained in the smooth locus of X by hypothesis. Normalization is an isomorphism over that locus, so the conductor locus is disjoint from $g(Y_0 \cap G)$. Under the open-chart identification, this locus is therefore contained in

$$Y_0 \setminus G.$$

This set has codimension at least two by hypothesis; it cannot contain the divisor just obtained. This contradiction proves that A is not an S_2 ring. A Cohen–Macaulay local ring has Cohen–Macaulay localizations [10, Tag 00NB] and hence is S_2, so A is not Cohen–Macaulay. If X were S_2, its localization A would be S_2 as well, giving the final assertion. □

Remark 3.4. Only the two specified immersive decompositions are used. Other decompositions, additional normalization points, and positive-dimensional fibers elsewhere cause no difficulty: Zariski's Main Theorem is applied only on the open quasi-finite locus. The argument also makes no claim that X is smooth in codimension one.

4 Integer witnesses in every rank at least five

4.1 Six rank-one matrices and a five-term tensor

Identify $\mathbb{C}^3 \otimes \mathbb{C}^3$ with 3×3 matrices, so that $a \otimes b$ corresponds to ab^{T}. Set

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}. \tag{4.1}$$

Let r_i be its ith row, m_j its jth column, and e_1, e_2, e_3 the coordinate vectors. Define

$$Q_i = e_i r_i \quad (1 \le i \le 3), \qquad Q_{3+j} = -m_j e_j^{\mathsf{T}} \quad (1 \le j \le 3). \tag{4.2}$$

Thus $Q_s = a_s b_s^{\mathsf{T}}$ for the following integer factors:

s	a_s^{T}	b_s^{T}
1	$(1,0,0)$	$(1,1,1)$
2	$(0,1,0)$	$(1,2,3)$
3	$(0,0,1)$	$(1,4,9)$
4	$(-1,-1,-1)$	$(1,0,0)$
5	$(-1,-2,-4)$	$(0,1,0)$
6	$(-1,-3,-9)$	$(0,0,1)$

They are nonzero and pairwise nonproportional: the row-supported and column-supported matrices have different supports, and all entries of M are nonzero.

Lemma 4.1. *The only linear relation among $Q_1,\dots,Q_6$, up to a scalar, is*

$$Q_1+Q_2+Q_3+Q_4+Q_5+Q_6=0. \tag{4.3}$$

Their span L has dimension five, and any five of these matrices form a basis of L. Every $N\in L$ has an expression

$$N_{ij}=M_{ij}(x_i-y_j),\qquad 1\le i,j\le 3. \tag{4.4}$$

Proof. Summing the first three matrices gives M, and summing the last three gives $-M$. If $\sum_s \alpha_s Q_s=0$, its (i,j) entry is $M_{ij}(\alpha_i-\alpha_{3+j})=0$. Every M_{ij} is nonzero, so all six coefficients coincide. The unique relation uses every matrix with a nonzero coefficient, proving independence of any five. Formula (4.4) is the linear combination with coefficients x_1,x_2,x_3,y_1,y_2,y_3 in (4.2). □

Let $f_1,\dots,f_5$ be the standard basis of $\mathbb{C}^5$. Our tensor is

$$T=\sum_{i=1}^{5} Q_i\otimes f_i\in\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^5. \tag{4.5}$$

This specifies all its integer entries; explicitly, its five matrix slices are

$$\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\1&2&3\\0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0\\0&0&0\\1&4&9\end{pmatrix},\quad \begin{pmatrix}-1&0&0\\-1&0&0\\-1&0&0\end{pmatrix},\quad \begin{pmatrix}0&-1&0\\0&-2&0\\0&-4&0\end{pmatrix}. \tag{4.6}$$

Relation (4.3) gives the second decomposition

$$T=\sum_{i=1}^{4} Q_i\otimes(f_i-f_5)-Q_6\otimes f_5. \tag{4.7}$$

In particular the final summand in (4.7) has factors $(1,3,9)^{\mathsf{T}}$, e_3, and f_5; the preceding four have the factors in the table and the indicated third vectors.

Each expression has five distinct nonzero pure summands. The two unordered sets are different even if one identifies a pure summand's different factorizations: the first uses the matrix direction Q_5, whereas the second uses Q_6, and these are distinct directions. In either expression the five third vectors are a basis of $\mathbb{C}^5$, and the five matrix factors are a basis of L.

The third flattening of T has image L and rank five. Every sum of four pure tensors has all flattening ranks at most four, and the 5×5 minors of a flattening vanish on the Zariski closure of that set. Consequently the displayed five-term decomposition and the flattening rank prove

$$\operatorname{rank}(T)=\underline{\mathrm{R}}(T)=5. \tag{4.8}$$

Here $\operatorname{rank}(T)$ in (4.8) denotes tensor rank; matrix ranks are specified as such elsewhere.

4.2 Six tangent intersections

Let $\mathcal{R}$ be the nonzero rank-one matrix cone in $\mathbb{C}^3 \otimes \mathbb{C}^3$. For $Q = ab^{\mathsf{T}}$ its affine tangent space is

$$T_Q\mathcal{R} = a(\mathbb{C}^3)^{\mathsf{T}} + \mathbb{C}^3 b^{\mathsf{T}}. \tag{4.9}$$

The formula follows by differentiating ab^{T}; its image has dimension five, with one-dimensional scaling kernel, and equals the tangent space of the smooth rank-one locus.

Lemma 4.2. *For each $s = 1, \dots, 6$,*

$$L \cap T_{Q_s}\mathcal{R} = \mathbb{C}Q_s. \tag{4.10}$$

Proof. Fix a row point $Q_i = e_i r_i$, and let $k < \ell$ be the two remaining row indices. By (4.9), tangency requires rows k and ℓ of N to be $z_k r_i$ and $z_\ell r_i$. If N also belongs to L, formula (4.4) gives

$$y_j = x_k - z_k \frac{M_{ij}}{M_{kj}} = x_\ell - z_\ell \frac{M_{ij}}{M_{\ell j}} \quad (j = 1, 2, 3).$$

The coefficient vectors in these three equations are independent. Indeed, define

$$D_i = \det \begin{pmatrix} 1 & 1 & 1 \\ M_{i1}/M_{k1} & M_{i2}/M_{k2} & M_{i3}/M_{k3} \\ M_{i1}/M_{\ell 1} & M_{i2}/M_{\ell 2} & M_{i3}/M_{\ell 3} \end{pmatrix}.$$

Direct rational expansion gives

$$(D_1, D_2, D_3) = \left(-\frac{1}{18}, \frac{1}{3}, -2\right). \tag{4.11}$$

Thus $x_k = x_\ell$ and $z_k = z_\ell = 0$. All y_j equal x_k, and (4.4) leaves only a scalar multiple of Q_i.

For a column point Q_{3+j}, the minus sign in (4.2) does not change the tangent space. Interchange rows and columns in the preceding argument. With $k < \ell$ the other two column indices, the required determinant is

$$E_j = \det \begin{pmatrix} 1 & 1 & 1 \\ M_{1j}/M_{1k} & M_{2j}/M_{2k} & M_{3j}/M_{3k} \\ M_{1j}/M_{1\ell} & M_{2j}/M_{2\ell} & M_{3j}/M_{3\ell} \end{pmatrix}.$$

Its values are

$$(E_1, E_2, E_3) = \left(-\frac{1}{18}, \frac{4}{9}, -\frac{3}{2}\right). \tag{4.12}$$

They are again nonzero. The same equations now force every column other than j to vanish and leave a scalar multiple of Q_{3+j}. This proves (4.10) in all six cases. □

4.3 From matrix tangencies to tensor immersion

Lemma 4.3. *Let $P_1, \dots, P_r$ be independent nonzero rank-one matrices in $A \otimes B$, put $L = \langle P_1, \dots, P_r \rangle$, and suppose*

$$L \cap T_{P_i}\mathcal{R} = \mathbb{C}P_i \quad (1 \le i \le r),$$

where $\mathcal{R}$ is the nonzero rank-one matrix cone. If $c_1, \dots, c_r$ are a basis of C, then addition on the product of the r nonzero pure-tensor cones has injective differential at $(P_i \otimes c_i)_{i=1}^r$.

Proof. Change the basis of C so that $c_i = f_i$. Write a tangent vector at the ith summand as

$$\dot{P}_i \otimes f_i + P_i \otimes \dot{c}_i, \qquad \dot{P}_i \in T_{P_i}\mathcal{R}, \qquad \dot{c}_i = \sum_j h_{ji} f_j.$$

Suppose that the sum of these r tangent vectors is zero. Comparing coefficients of f_j gives

$$\dot{P}_j + \sum_i h_{ji} P_i = 0.$$

Thus $\dot{P}_j \in L \cap T_{P_j}\mathcal{R} = \mathbb{C}P_j$; write $\dot{P}_j = \lambda_j P_j$. Independence of the P_i now implies $h_{ji} = 0$ for $i \neq j$ and $h_{jj} = -\lambda_j$. Consequently the original tangent vector at each individual summand is zero. The tangent spaces of the pure-tensor cones therefore sum directly. This argument works intrinsically on the pure-tensor cones, after removing the factor-coordinate scaling directions. □

Applying the lemma to each decomposition of $T_5 := T$ in (4.5) and (4.7) gives differential rank

$$5(3+3+5-2) = 45. \tag{4.13}$$

The sign change from Q_6 to $-Q_6$ does not change its tangent space.

4.4 Two avoidance identities

Write $A_0 = B_0 = \mathbb{C}^3$, and denote the span of the six matrices by $L_0 \subset A_0 \otimes B_0$. For an integer $t \geq 4$ put

$$v(t) = (1, t, t^2)^{\mathsf{T}}.$$

Lemma 4.4. *For every integer $t \geq 4$,*

$$L_0 \cap \big(\mathbb{C}v(t) \otimes B_0\big) = 0, \qquad L_0 \cap \big(A_0 \otimes \mathbb{C}v(t)\big) = 0. \tag{4.14}$$

Moreover, the lines $\mathbb{C}v(t)$ are pairwise distinct and avoid all first- and second-factor lines in the six core matrices.

Proof. Since $M_{ij} = j^{i-1}$, every matrix in L_0 has entries $N_{ij} = j^{i-1}(x_i - y_j)$. First suppose that $N = v(t)b^{\mathsf{T}}$. The first row gives $y_j = x_1 - b_j$. Substitution into the second row, with $d = x_1 - x_2$, yields

$$b_j = \frac{jd}{j-t} \qquad (j = 1, 2, 3).$$

All denominators are nonzero. Substitution into the third row, with $e = x_1 - x_3$, gives

$$(j+t)d = je \qquad (j = 1, 2, 3).$$

The equations for $j = 1, 2$ imply $td = 0$. Thus $d = e = 0$ and $b = 0$, which proves the first intersection statement.

Now suppose $N = av(t)^{\mathsf{T}}$. Let $\mathbf{1} = (1, 1, 1)$ and $y = (y_1, y_2, y_3)$. Row i says

$$y = x_i\mathbf{1} - a_i\frac{v(t)^{\mathsf{T}}}{r_i},$$

with entrywise division. The vectors $\mathbf{1}$, $v(t)^{\mathsf{T}}$, and $v(t)^{\mathsf{T}}/(1, 2, 3)$ are independent, since

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & t & t^2 \\ 1 & t/2 & t^2/3 \end{pmatrix} = -\frac{t(t-1)(t-3)}{6} \neq 0. \tag{4.15}$$

The planes obtained from rows one and two therefore meet only in $\mathbb{C}\mathbf{1}$. Hence $a_1 = a_2 = 0$ and y is constant. The vector $(1, t/4, t^2/9)$ from row three is not constant: constancy would require simultaneously $t = 4$ and $t^2 = 9$. Thus $a_3 = 0$ as well, proving the second statement.

Finally, the first coordinate of every $v(t)$ is one, so distinct integers t give distinct lines. None is a coordinate line. The other core first-factor lines are those of $v(1), v(2), v(3)$. The other second-factor vectors are $(1, 1, 1)$, $(1, 2, 3)$ and $(1, 4, 9)$. Direct comparison with $(1, t, t^2)$ for $t \geq 4$ excludes every such line. □

4.5 Adding rows and columns

Fix $r \geq 5$ and integers $p, q \geq 3$ with $p + q = r + 1$. Set

$$A' = A_0 \oplus E = \mathbb{C}^p, \qquad B' = B_0 \oplus F = \mathbb{C}^q,$$

where A_0, B_0 occupy the first three coordinates. Let $u_1, \ldots, u_{p-3}$ and $w_1, \ldots, w_{q-3}$ be the coordinate bases of E and F. Define the following integer rank-one matrices:

$$R_s = u_s \otimes v(s+3) \quad (1 \leq s \leq p-3), \qquad S_t = v(t+3) \otimes w_t \quad (1 \leq t \leq q-3). \tag{4.16}$$

Empty lists are permitted. Their span with the core is the direct sum

$$L = L_0 \oplus \langle R_1, \ldots, R_{p-3}\rangle \oplus \langle S_1, \ldots, S_{q-3}\rangle. \tag{4.17}$$

Indeed, L_0 lies in $A_0 \otimes B_0$, the R_s lie in distinct rows of $E \otimes B_0$, and the S_t lie in distinct columns of $A_0 \otimes F$. In particular $\dim L = 5 + (p-3) + (q-3) = r$.

Lemma 4.5. *The intersection of L with the affine rank-one matrix tangent space at each Q_i, R_s, and S_t is precisely the scalar line of that matrix.*

Proof. Use the four blocks given by $(A_0 \oplus E) \otimes (B_0 \oplus F)$. The space L has no $E \otimes F$ block. At a core matrix $Q_i = a_i \otimes b_i$, every extra row of a tangent matrix is proportional to b_i. In L, its row indexed by u_s is proportional to $v(s+3)$. These lines differ by Lemma 4.4, so all coefficients of the R_s vanish. In the same way, the extra columns of a tangent matrix are proportional to a_i, whereas those of L are proportional to the corresponding $v(t+3)$. All coefficients of the S_t vanish. The remaining intersection in the core block is $\mathbb{C}Q_i$ by Lemma 4.2.

At $R_s = u_s \otimes v(s+3)$, write a tangent matrix as

$$x \otimes v(s+3) + u_s \otimes y.$$

Its $E \otimes F$ block forces the F-part of y to be zero if this matrix lies in L. Its possible F-columns are supported on u_s, whereas the F-columns of L are supported on A_0; hence all S_t coefficients vanish. For an E-row indexed by u_j with $j \neq s$, the tangent row is proportional to $v(s+3)$ while the L-row is proportional to $v(j+3)$. Their distinct lines force every other R_j coefficient to be zero. The core block is an element of $L_0 \cap (A_0 \otimes \mathbb{C}v(s+3))$, which is zero by (4.14). Only the row indexed by u_s remains, and its membership in L makes it a scalar multiple of R_s.

At $S_t = v(t+3) \otimes w_t$, interchange rows and columns in this argument. The $E \otimes F$ block vanishes, the other extra columns are excluded by distinctness of the v-lines, and the core block is in $L_0 \cap (\mathbb{C}v(t+3) \otimes B_0) = 0$. The intersection is $\mathbb{C}S_t$. These comparisons also cover the case when both E and F are nonzero. □

Let $f_1, \ldots, f_r$ be the coordinate basis of $\mathbb{C}^r$. The required integer tensor in the seed format $p \times q \times r$ is

$$T_{r;p,q} = \sum_{i=1}^{5} Q_i \otimes f_i + \sum_{s=1}^{p-3} R_s \otimes f_{5+s} + \sum_{t=1}^{q-3} S_t \otimes f_{p+2+t}. \tag{4.18}$$

Replacing its first five terms by

$$\sum_{i=1}^{4} Q_i \otimes (f_i - f_5) - Q_6 \otimes f_5 \tag{4.19}$$

and leaving the added terms unchanged gives its second decomposition. All entries and all displayed decomposition factors are integers.

Proposition 4.6. *Let $r \geq 5$, $p, q \geq 3$, and $p + q = r + 1$. For any $a \geq p$, $b \geq q$, $c \geq r$, pad $T_{r;p,q}$ with zero coordinates to format $a \times b \times c$. Its ordinary rank and border rank are both r. The two displayed unordered r-term decompositions are distinct, have independent third-factor vectors, and have injective addition differentials. Each differential has rank $r(a + b + c - 2)$.*

Proof. In either decomposition, the r matrix factors form a basis of L: any five core matrices form a basis of L_0, and the added matrices are independent by (4.17). The r third factors are also a basis. In the second decomposition this follows from the invertible change replacing f_i by $f_i - f_5$ for $1 \leq i \leq 4$, with f_5 and the remaining basis vectors retained. The choice of Q_5 in the first decomposition and Q_6 in the second makes the unordered pure summands distinct between the two decompositions. Each summand is nonzero; independence of the third factors makes summands within a decomposition distinct.

Lemma 4.5 and Lemma 4.3 prove immersion in the seed format. The third flattening has image L and rank r. Flattening minors vanish on the border-rank-at-most-$(r-1)$ variety, while the displayed decomposition has r terms. Thus both tensor rank and border rank equal r.

Choose complements for the three inclusions into the padded spaces. The derivative directions obtained by varying the first factor in its complement are

$$(\mathbb{C}^a / A') \otimes \langle b_i \otimes c_i : 1 \leq i \leq r \rangle,$$

with an analogous expression for the second factor; we identify the quotient with the chosen complement. Since the c_i are independent and every a_i, b_i is nonzero, both opposite-factor pair lists are independent. Variation of the third factor in its complement gives $L \otimes (\mathbb{C}^c / \mathbb{C}^r)$, also with independent matrix factors. These three

spaces occupy different complementary tensor blocks and are disjoint from the original derivative image. Their dimensions are $r(a-p)$, $r(b-q)$, and $r(c-r)$. Adding them to the seed rank gives

$$r(p+q+r-2)+r(a-p)+r(b-q)+r(c-r)=r(a+b+c-2).$$

This is the dimension of the product of the r pure-tensor cones in the padded format. Hence its differential is injective. The free finite permutation quotient is etale, so the unordered differential is injective as well. Padding preserves the flattening rank and the displayed tensor equality, proving all remaining assertions. □

For the hypotheses of Theorem 1.1, choose explicitly

$$p=\max\{3,r+1-b\}, \qquad q=r+1-p. \tag{4.20}$$

Then $p\le a$ because $a\ge 3$ and $a+b\ge r+2$; also $p\le r-2$ because $r\ge 5$ and $b\ge 3$. It follows that $q\ge 3$ and $q\le b$. Thus (4.18) specifies an integer witness for every format in the main theorem. At $r=5$ the seed is the original $3\times 3\times 5$ tensor. Its two differential ranks after padding are 50 in format $3\times 4\times 5$ and 80 in format $6\times 6\times 6$.

5 The depth defect and the projective statement

Proof of Theorem 1.1. Apply Proposition 2.5 with the dimensions in (1.2). It gives a smooth irreducible source Y of unordered decompositions with independent third factors, a birational addition morphism $f: Y\to X$, and an open $G\subseteq Y$ whose image is smooth and whose complement has codimension at least two. It also gives $\dim X = D = r(a+b+c-2)$.

Choose p,q by (4.20) and pad the integer tensor in (4.18) to the target format. Proposition 4.6 supplies two distinct points of Y over this tensor T at which df is injective. Proposition 3.3 therefore proves that $\mathcal{O}_{X,T}$ is not an S_2 ring and is not Cohen–Macaulay. Consequently X itself fails S_2. The witness has ordinary rank and border rank r by the same construction.

It remains to prove the precise depth assertion at the homogeneous vertex. Put $N=abc$, let $S=\mathbb{C}[z_1,\dots,z_N]$, and write $R=S/J$, where $J=I(X)$ is a homogeneous prime ideal. Then $\operatorname{ht} J = N-D$. Suppose that $R_{\mathfrak{m}}$ were Cohen–Macaulay, and put $\mathfrak{n}=(z_1,\dots,z_N)$ in S. The Auslander–Buchsbaum formula over the regular local ring $S_{\mathfrak{n}}$ gives

$$\operatorname{pd}_{S_{\mathfrak{n}}} R_{\mathfrak{m}} = N-D \qquad \text{[10, Tag 090V]}.$$

A polynomial ring has finite global dimension [10, Tag 00OQ], so R has a finite minimal graded free resolution over S. The entries in each differential lie in $\mathfrak{n}$, hence its localization at $\mathfrak{n}$ is still minimal. Every nonzero free term remains nonzero there. The graded resolution therefore has length exactly $N-D$.

For any prime $\mathfrak{P}\supseteq J$, localizing this resolution gives

$$\operatorname{pd}_{S_{\mathfrak{P}}} R_{\mathfrak{P}/J} \le N-D.$$

Regularity of $S_{\mathfrak{P}}$ and Auslander–Buchsbaum yield

$$\begin{aligned}\operatorname{depth} R_{\mathfrak{P}/J} &= \dim S_{\mathfrak{P}} - \operatorname{pd}_{S_{\mathfrak{P}}} R_{\mathfrak{P}/J}\\ &\ge \operatorname{ht}\mathfrak{P} - (N-D) = \dim R_{\mathfrak{P}/J}.\end{aligned}$$

The last equality is the height formula for a prime quotient of a polynomial ring. Since depth cannot exceed dimension, every local ring of R would be Cohen–Macaulay. This contradicts the local ring at T. Hence $\operatorname{depth} R_{\mathfrak{m}} < D$. Permuting tensor factors is a linear isomorphism preserving the varieties and their local rings, proving the final assertion. □

Proof of Corollary 1.2. The stated formats satisfy (1.2), and substitution in $D=r(a+b+c-2)$ gives the displayed dimensions and depth bounds. For the orbit identity, the dense open set of triples of invertible $r\times r$ factor matrices parametrizes precisely the $(\mathrm{GL}_r)^3$ orbit of Δ_r. Allowing arbitrary factor matrices parametrizes all sums of r pure tensors in format $r\times r\times r$. The image closures of a morphism and of its restriction to a dense open of its irreducible source are equal. This proves (1.4) and its asserted consequences. □

Proof of Corollary 1.3. The witness T is nonzero. Choose a degree-one coordinate ℓ with $\ell(T) \neq 0$. Scalar rescaling gives an isomorphism

$$X \cap D(\ell) \simeq D_+(\ell) \times \mathbb{G}_m,$$

where $D_+(\ell)$ is the corresponding affine chart of $\operatorname{Proj} R$. Explicitly, a tensor is sent to its representative with $\ell = 1$ and to its nonzero ℓ-value. At the selected closed point, the local ring on the right is a localized Laurent-polynomial extension of the projective local ring. Cohen–Macaulayness ascends to this extension. Conversely, quotienting by the regular parameter $t-\ell(T)$ recovers the projective local ring and preserves Cohen–Macaulayness. Thus the two local rings are Cohen–Macaulay simultaneously. The affine local ring at T is not Cohen–Macaulay, so the projective local ring at $[T]$ is not Cohen–Macaulay either. Its arithmetic Cohen–Macaulayness is already excluded by (1.3). □

6 Boundaries of the decomposition obstruction

The existence of two immersive decompositions has a sharp rank threshold for tensors with three factors. It also behaves differently from the Cohen–Macaulay property under a change of format. We establish these points before discussing the relation with multiplication tensors. Throughout this section, immersion means injectivity of the addition differential on the space of unordered decompositions into distinct nonzero pure summands.

6.1 The first possible rank

Theorem 6.1. *Let A, B, C be finite-dimensional complex vector spaces and let $1 \leq r \leq 4$. No tensor in $A \otimes B \otimes C$ has two distinct immersive unordered decompositions into r nonzero pure summands. Consequently, among three-factor tensors in arbitrary finite dimensions, rank five is the first rank admitting this decomposition obstruction.*

Proof. The assertion for $r = 1$ is immediate. Suppose $2 \leq r \leq 4$ and

$$T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i$$

is an immersive decomposition. Each list of opposite-factor products is linearly independent. For example, a nonzero relation $\sum_i \lambda_i b_i \otimes c_i = 0$ and a fixed $0 \neq z \in A$ would give the nonzero source tangent vector with components $\lambda_i z \otimes b_i \otimes c_i$, whose image under the addition differential is zero. The other two lists are treated in the same way.

It follows that the rank of each factor-column matrix equals the corresponding flattening rank of T. Indeed, the first flattening is the product of the first factor-column matrix and the transpose of the full-column-rank matrix of opposite products. The latter transpose is surjective, so the product has the same image as the former matrix. We may therefore restrict all three factors to the supports of T; all summands lie in these supports, and any tangent kernel after restriction would also be a tangent kernel in the original format. Write their dimensions as a, b, c.

If $a, b, c \leq r - 1$, the source dimension is $r(a + b + c - 2)$, which is strictly larger than the ambient dimension abc. To check this for all positive a, b, c in the stated range, observe that

$$\frac{abc}{a + b + c - 2}$$

is nondecreasing in each variable: its derivative with respect to a is $bc(b + c - 2)/(a + b + c - 2)^2 \geq 0$. Its maximum is $(r - 1)^3/(3r - 5)$, whose values for $r = 2, 3, 4$ are respectively $1, 2, 27/7$, all strictly smaller than r. Immersion is thus impossible unless at least one flattening has rank r.

Permute the factors so that $c_1, \ldots, c_r$ are independent, and put $P_i = a_i \otimes b_i$. The C-flattening has image $L = \langle P_1, \ldots, P_r \rangle$, with the P_i independent. Let $\mathcal{R} \subset A \otimes B$ be the rank-at-most-one matrix cone. Immersion implies

$$L \cap T_{P_i}\mathcal{R} = \mathbb{C}P_i \qquad (1 \leq i \leq r). \tag{6.1}$$

Indeed, suppose $\delta P_i \in L \cap T_{P_i}\mathcal{R}$ is not proportional to P_i, and write $\delta P_i = \sum_j h_j P_j$. Cancel $\delta P_i \otimes c_i$ by the tangent variations $-h_j P_j \otimes c_i$ in the jth summands. The ith component of the resulting source tangent vector is $(\delta P_i - h_i P_i) \otimes c_i \neq 0$, while its sum is zero, contradicting immersion.

Any other r-term decomposition has independent C-factors and independent matrix factors spanning the same L, because the C-flattening has rank r. If its matrix-factor directions are the original $[P_i]$, comparison in this basis fixes the pure summands. Otherwise it supplies a rank-one matrix $Q \in L$ whose direction is different from every $[P_i]$.

The expansion of Q in the basis $P_1, \dots, P_r$, restricted to its nonzero coefficients, gives a minimally dependent set of m rank-one matrices, where $3 \le m \le r+1 \le 5$. Its span L_0 has dimension $m-1$, and every coefficient in its unique relation is nonzero. Let U, V be the spans of their first and second factors, of dimensions u, v. Writing the relation as

$$U_{\mathrm{col}} D V_{\mathrm{col}}^{\mathsf{t}} = 0,$$

where D is an invertible diagonal $m \times m$ matrix, shows that $u + v \le m$: the rank-v image of $DV_{\mathrm{col}}^{\mathsf{t}}$ is contained in the $(m-u)$-dimensional kernel of U_{col}.

If $u = 1$ or $v = 1$, the entire L_0 lies in the rank-one-cone tangent at each of its nonzero rank-one points. Since $\dim L_0 = m - 1 \ge 2$, this contradicts (6.1) at an original P_i in the relation. Otherwise $u, v \ge 2$ and $u + v \le 5$. Up to exchange, the only possibilities are $(2,2)$ and $(2,3)$. The affine tangent to the rank-one cone in $U \otimes V$ has dimension $u + v - 1$, so at any point P in the relation,

$$\dim\bigl(L_0 \cap T_P\mathcal{R}\bigr) \ge (m-1) + (u+v-1) - uv.$$

For $(u,v) = (2,2)$, we have $m \ge 4$ and this lower bound is $m - 2 \ge 2$. For $(2,3)$, we have $m \ge 5$ and it is $m - 3 \ge 2$. Taking an original P_i again contradicts (6.1). Thus the immersive decomposition is unique. The rank-five example in Proposition 4.6 proves that the bound is sharp. □

Theorem 6.1 concerns this particular source of normalization branches. It does not assert that every secant of rank at most four is Cohen–Macaulay, or that rank five is the least rank for failure of arithmetic Cohen–Macaulayness by any method.

6.2 A Cohen–Macaulay adjacent family

Proposition 6.2. *For every integer $c \ge 1$, the variety $X_5(3,3,c)$ is Cohen–Macaulay. For $c \le 5$ it is the entire ambient tensor space. For $c \ge 5$, third flattening identifies it, with reduced structure, with the variety of $9 \times c$ matrices of rank at most five.*

Proof. The core decomposition of Lemma 4.1 has addition differential rank 45 in format $3\times3\times5$, by Proposition 4.6. A nonzero 45×45 derivative minor implies that the image closure has dimension at least 45. This is the dimension of the ambient space, so $X_5(3,3,5)$ is that entire space. Projection of the third factor to $\mathbb{C}^c$, for $1 \le c \le 5$, proves the corresponding assertion there: a linear map preserves sums of five pure tensors and cannot increase border rank.

Now let $c \ge 5$. Third flattening puts $X_5(3,3,c)$ inside the irreducible determinantal variety D of $9 \times c$ matrices of rank at most five, whose dimension is $5(9 + c - 5) = 5(c+4)$. Pad the core decomposition in its third factor. Each of the $c-5$ new third-factor directions contributes the five independent matrices $P_1, \dots, P_5$ to the differential image. These new blocks have distinct third-factor coordinates and are independent of the original image. Thus the addition differential has rank

$$45 + 5(c-5) = 5(c+4).$$

The secant and D consequently have the same dimension. Since the secant is a closed irreducible subvariety of the irreducible D, they are equal as reduced varieties. For $c \ge 6$ the Cohen–Macaulay assertion is the generic determinantal case; in this format it also follows from [3, Theorem 2.4]. For $c = 5$ the determinantal condition is vacuous. □

Thus, at fixed rank bound five, first-factor dimension three, and $c \ge 5$, increasing the second-factor dimension from three to four changes a Cohen–Macaulay variety to a non-S_2 variety. This statement does not establish minimality among all formats. In particular, the non-S_2 result is not a consequence of merely placing the same two decompositions inside an arbitrary larger tensor space.

6.3 An obstruction on invertible-slice charts

For $T \in A \otimes B \otimes C$ and $\alpha \in A^*$, write $T_\alpha : B^* \to C$ for the contraction with α.

Proposition 6.3. *Let $r \geq 1$, let $\dim B = \dim C = r$, and suppose that $T \in A \otimes B \otimes C$ has an invertible slice T_α. If T has a decomposition into r nonzero pure summands, then either that unordered decomposition is unique, or every such decomposition lies in a positive-dimensional family and has a nonzero kernel for the addition differential. In particular, two distinct immersive r-term decompositions cannot occur on this chart. No conciseness assumption on the A factor is required.*

Proof. In an r-term decomposition $T = \sum_i a_i \otimes b_i \otimes c_i$, invertibility of T_α forces every $\alpha(a_i)$ to be nonzero and both (b_i) and (c_i) to be bases. Absorb the scalar $\alpha(a_i)$ into b_i so that $\alpha(a_i) = 1$. For each $\beta \in A^*$, the endomorphism

$$X_\beta = T_\beta T_\alpha^{-1} : C \longrightarrow C$$

is diagonal in the basis (c_i), with eigenvalues $\beta(a_i)$. The joint eigenspaces are

$$E_a = \langle c_i : a_i = a \rangle.$$

Both their dimensions and their labels $\beta \mapsto \beta(a)$ are intrinsic to T. If the a_i are distinct, every joint eigenspace is a line. These lines determine the c_i directions, and T_α determines their paired b_i coefficients. The labels then fix the normalized a_i, proving uniqueness of the pure summands.

If $a_1 = a_2 = a$, replace the first two summands by

$$a \otimes (b_1 + tb_2) \otimes c_1 + a \otimes b_2 \otimes (c_2 - tc_1).$$

Their sum is unchanged. The two altered factor lists remain bases, so the summands remain nonzero and distinct. At $t = 0$ the two source tangent components are respectively $a \otimes b_2 \otimes c_1$ and its negative. They are nonzero and their sum is zero. Passing to the free finite permutation quotient does not remove this tangent kernel. The curve is nonconstant modulo permutations and therefore gives a positive-dimensional family. An intrinsic joint eigenspace of dimension at least two has repeated labels in every decomposition, so this argument applies to every one. □

This proposition applies only to tensors admitting r actual pure summands. It makes no assertion about the Cohen–Macaulay property of the whole invertible-slice locus, including tensors of ordinary rank larger than r.

6.4 Concise lifts and the Hilbert–Quot comparison

A tensor is *concise* if each of its flattenings has full rank in its factor. The core tensor T_5 in (4.5) has flattening ranks $(3, 3, 5)$: the third rank follows from independence of the five matrices, while the first two follow from the three coordinate row directions and the independent rows of M, with $\det M = 2$. Hence every slice in any factor has matrix rank at most three. Padding, invertible changes of basis, and factor permutations do not change this conclusion. In particular, this tensor has no invertible slice between two five-dimensional factors.

There is also a direct description of its lifts under the concise secant construction of [5]. Let $c\sigma_5$ denote the projective concise secant variety in the ambient format containing T_5.

Proposition 6.4. *The two displayed decompositions of T_5 determine distinct smooth points of $c\sigma_5$ above $[T_5]$. With suitable framings, the concise tensor at each point is the unit tensor $\Delta_5 = \sum_{i=1}^5 e_i \otimes e_i \otimes e_i$.*

Proof. Write either decomposition as $T_5 = \sum_i a_i \otimes b_i \otimes c_i$ in $V_1 \otimes V_2 \otimes V_3$. Set $W_1 = W_2 = W_3 = \mathbb{C}^5$ and define

$$\widetilde{T} = \Delta_5, \qquad \phi_1(e_i) = a_i, \quad \phi_2(e_i) = b_i, \quad \phi_3(e_i) = c_i.$$

These data satisfy the intermediate conciseness conditions of [5, Proposition 3.20]. The tensor $\widetilde{T}$ is concise; after applying ϕ_3, the first two flattenings have rank five, because the remaining coordinate basis vectors separate the nonzero summands. After applying ϕ_2 and ϕ_3, the first flattening has rank five because the products $b_i \otimes c_i$ are independent: the c_i form a basis of their support. The final restriction is T_5.

In the projective framed construction, these data lie in an open subset of

$$X_5(5,5,5)^{\mathrm{conc}} \times \mathbb{P}\operatorname{Hom}(W_1, V_1) \times \operatorname{Hom}(W_2, V_2) \times \operatorname{Hom}(W_3, V_3). \tag{6.2}$$

Here the first restriction map is projectivized. The superscript conc denotes the concise locus of the *affine* secant. Apart from the equations defining that affine secant, the conditions are open; see [5, Proposition 3.20 and Theorem 3.25].

The unit tensor is smooth in $X_5(5,5,5)$. Indeed, decompositions whose three factor lists are bases form a dense open of the source, and their image is exactly the GL_5^3 orbit of Δ_5. This orbit is dense and locally closed, hence open; an algebraic group orbit over $\mathbb{C}$ is smooth. Thus the product (6.2) is smooth at both framed points. The map from the framed concise secant to $c\sigma_5$ is a principal GL_5^3 bundle, so smoothness descends to both unframed points.

To see that the two points are distinct, note that Δ_5 has a unique five-term decomposition: apply Proposition 6.3 with the functional taking value one on every e_i; its five normalized first-factor labels are distinct. An identification of the two framed data up to GL_5^3 would therefore permute these five pure summands. Its images would make the two original decompositions equal, possibly after a common scalar arising from $[\phi_1]$. That scalar is one since both sums equal the same nonzero T_5. The original decompositions are distinct, so no such identification exists. □

Under the standard basis identifications, Δ_5 is the multiplication tensor of the semisimple algebra $\mathbb{C}^5$. Embedded reduced configurations of five distinct points are smooth points of a Hilbert scheme of points on a smooth variety: their locus is the free symmetric-group quotient of the smooth configuration space of five ordered distinct points. Thus the algebra arising from these lifts supplies smooth Hilbert points. The two rank-one configurations themselves are different points of the Hilbert scheme of the Segre product; the map to the tensor secant forgets that configuration. In the rank-five formats of Theorem 1.1, the local Cohen–Macaulay failure of their common image does not transfer through this description to either configuration space.

For clarity, the conjectural consequences discussed in the literature must be distinguished from this local statement. In [5, §3.6.1, following Conjecture 3.32], universal arithmetic Cohen–Macaulayness of Segre secants is related, via smooth equivalences, to Cohen–Macaulayness assertions for surface Quot schemes and smoothable Hilbert components. The implications stated there run from the universal secant assertion; its failure supplies no negation of those assertions.

Haiman's original Conjecture 5.2.1 in [4] asks for *normality and Cohen–Macaulayness of the isospectral Hilbert scheme* over the principal component of $\mathrm{Hilb}^n(\mathbb{C}^d)$, for every n, d. The following paragraph observes that normality and Cohen–Macaulayness of the principal component follow by taking the symmetric-group quotient. Consequently, Cohen–Macaulayness of the smoothable Hilbert component is a weaker assertion than the original conjecture.

There is also a scheme-structure distinction on the Quot side. Artin–Hochster's conjecture, as stated in [9, Conjecture 3.2], concerns the commuting *scheme* cut out by the entries of $XY - YX$ for two square matrices. The presentation of surface Quot schemes in [9, Theorem 3.2] uses a stable open subset of this scheme with framing vectors and a principal general-linear-group quotient. By contrast, the minimal-border-rank locus defining the concise secant is explicitly reduced [5, Definition 3.14(a)]. Agreement of closed points alone cannot identify these scheme structures. Propositions 6.3 and 6.4 therefore give a limitation of the present decomposition obstruction; they provide neither a proof nor a counterexample to the Hilbert, isospectral Hilbert, or Quot Cohen–Macaulay assertions.

7 Normalization fibres and local structure

We now describe the entire normalization fibres at several of the explicit tensors above, the completed local ring at the rank-five witness, and the local models of specified higher-rank boundary points. Normalization is always taken in the function field of the reduced integral variety. The assertions about neighbourhoods and local rings below concern the displayed witnesses. They do not assert global smoothness, Cohen–Macaulayness, or rational singularities of a normalization. All conclusions transport under a permutation of the three tensor factors together with the specified witness.

7.1 A finite incidence model for normalization

The following criterion also applies beyond matrix Segre varieties.

Proposition 7.1 (Finite reduced intersection criterion)**.** *Let $S \subseteq \mathbb{P}W$ be a smooth irreducible projective variety, $r \geq 1$, and $\mathcal{U} \subseteq S^r$ the nonempty open set of ordered independent r-tuples. Let $H \subseteq \mathrm{Gr}(r, W)$ be the reduced closure of their spans, and suppose $\mathcal{U}/\mathfrak{S}_r \to H$ is birational. Suppose that $L \in H$ has scheme-theoretic intersection*

$$\mathbb{P}L \cap S = \{p_1, \ldots, p_m\}, \qquad m \geq r,$$

consisting of distinct reduced points, every r of which are independent. Then an open neighbourhood N of L has smooth normalization. The normalization morphism over N is unramified, and its fibre at L consists of exactly $\binom{m}{r}$ reduced points, indexed by the r-element subsets of the intersection.

Let C have dimension $c \geq r$, and let $X \subseteq W \otimes C$ be the reduced closure of sums $\sum_{i=1}^{r} P_i \otimes c_i$ with $[P_i] \in S$. If $T \in X$ has flattening rank r and image L as above, then an open neighbourhood of T has smooth normalization with unramified normalization morphism and exactly $\binom{m}{r}$ reduced points above T.

Proof. The incidence

$$Z = \{(K, p) \in H \times S : p \in \mathbb{P}K\} \longrightarrow H$$

is projective. Its fibre at L is finite, so upper semicontinuity of fibre dimension, followed by proper quasi-finite implies finite, makes it finite after shrinking about L. Its fibre cotangent spaces at the points above L vanish. Base change for relative differentials and Nakayama's lemma imply that the relative differentials vanish there. The residue extensions are trivial, so the morphism is unramified there. The image of its closed non-unramified locus is closed by finiteness. Shrinking once more, we obtain a finite unramified morphism $Z_N \to N$. No flatness is asserted.

In the r-fold fibre product of Z_N over N, the locus where two coordinates agree is open and closed: this follows from the diagonal criterion for separated unramified morphisms. The ordered distinct tuple locus D is consequently open and closed, and is finite unramified over N. Independence is open in D, and its closed complement has no point above L. Removing the finite image of that complement makes all tuples independent. Taking spans now identifies D with the open subset $s^{-1}(N)$ of $\mathcal{U}$. These are inverse morphisms on families, since the direct sum of r independent tautological lines in a rank-r subbundle is that subbundle. In particular D is smooth and irreducible.

The free permutation action gives a finite *étale* quotient $D \to E = D/\mathfrak{S}_r$; over affine opens of N the quotient is given by invariants in a finite algebra. Thus E is smooth and integral, and $E \to N$ is finite and birational. It is the normalization: a finite normal extension of domains with the same fraction field equals the integral closure. Unramifiedness descends through $D \to E$. The fibre consists of unordered r-subsets; since the map is unramified and the residue field is $\mathbb{C}$, the fibre is reduced. This proves the first assertion.

The rank-r locus of X is the bundle of surjections $C^* \to K$ over H. The proof of Lemma 2.4 applies unchanged: above an actual span all surjections are sums of the specified type, and the bundle projection is smooth and open, so taking closures gives the equality. Pull this bundle back to E. The result is smooth and integral, and maps finitely and birationally to the original bundle over N. It is therefore its normalization, and unramifiedness follows by base change. More generally, even when the normalization of H is singular, the normalization of the rank-r tensor bundle is its pullback to H's normalization: a vector bundle over a normal variety and every open subset of it are normal, and the map is finite birational. □

7.2 The core and the row–column configurations

Use M, Q_i, L_0, T_5 from (4.1), (4.2), and (4.5). Thus $T_5 = \sum_{i=1}^{5} Q_i \otimes f_i$.

Lemma 7.2. *The scheme $\mathbb{P}L_0 \cap \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2)$ consists exactly of the six reduced points $[Q_1], \ldots, [Q_6]$. The same assertion holds after padding the two matrix factors.*

Proof. By (4.4), membership of a matrix N in L_0 means $N_{ij} = M_{ij}(x_i - y_j)$. Suppose $N = ab^{\mathsf{T}}$ has rank one. If $a_i = 0$ for some i, its zero row gives $y_1 = y_2 = y_3 = x_i$. Every nonzero row is then proportional to the corresponding row of M. Since the rows of M are independent, rank one permits exactly one such row, giving one of Q_1, Q_2, Q_3. If any $b_j = 0$, the transposed argument gives one of Q_4, Q_5, Q_6.

If all coordinates are nonzero, choose projective representatives $a = (1, u, v)$ and $b = (1, s, t)$, where $u, v, s, t \neq 0$. The four additive rectangle relations for N_{ij}/M_{ij} are

$$E_1 = (u-2)s - 2(u-1) = 0, \qquad E_2 = (u-3)t - 3(u-1) = 0,$$
$$E_3 = (v-4)s - 4(v-1) = 0, \qquad E_4 = (v-9)t - 9(v-1) = 0.$$

Direct expansion gives the polynomial identity

$$3(u-4)\big((u-2)E_3 - (v-4)E_1\big) - (u-3)\big((u-3)E_4 - (v-9)E_2\big) = 6u(u-1).$$

It forces $u = 1$, and then $E_1 = 0$ forces $s = 0$, a contradiction. This exhausts the rank-one points. Lemma 4.2 gives zero projective tangent space at each of them. Nakayama's lemma, applied to the local maximal ideal modulo its square, makes each local ring $\mathbb{C}$, proving reducedness.

A nonzero rank-one tensor contained in $A_0 \otimes B_0$ has its two factors in A_0 and B_0: apply the quotient map by either subspace and use that the other factor is nonzero. Thus padding creates no new points. Complementary factor variations occupy different tensor blocks and cannot contribute to a tangent vector in L_0. The tangent intersections, and hence reducedness, are also preserved. □

Corollary 7.3. *For $a, b \geq 3$, $a + b \geq 7$, and $c \geq 5$, the normalization of $X_5(a, b, c)$ has exactly six reduced points above the padded T_5. They correspond bijectively to its six actual unordered five-term decompositions, all immersive. On an open neighbourhood of T_5 the entire normalization is smooth and the normalization morphism is unramified.*

Proof. The subset rank criterion of Lemmas 2.1 and 2.2 makes the unordered span morphism birational in these dimensions. Lemma 4.1 says that every five of the six intersection points are independent. Apply Proposition 7.1 with $m = 6$ and $r = 5$. An actual five-term decomposition must give a basis of the flattening image L_0. Conversely, every such basis determines unique third coefficients for T_5, and these coefficients are independent because its flattening has rank five. Thus the six subsets give exactly six unordered decompositions. The scalar-only tangent intersections and Lemma 4.3, with the complementary-block argument of Proposition 4.6, prove immersion. □

Proposition 7.4. *Let $r \geq 5$, $p, q \geq 3$, and $p + q = r + 1$. For the span $L_r = L$ in (4.17), the scheme-theoretic rank-one intersection consists exactly of its $r + 1$ displayed points $[Q_i], [R_s], [S_t]$, all reduced. This holds after padding to any $a \geq p, b \geq q$. In every format $a \geq p, b \geq q, c \geq r$, the tensor $T_{r;p,q}$ has exactly six actual unordered r-term decompositions, all immersive, and has rank and border rank r.*

If also $a, b \geq 3$ and $a + b \geq r + 2$, its normalization fibre has at least $r + 1$ distinct points. Six distinguished smooth points are the actual decompositions. For $r \geq 6$, at least $r - 5$ additional points are required by dependent configurations. This lower bound alone makes no assertion about their smoothness or the exact total size of the fibre.

Proof. Write a nonzero rank-one matrix in L_r as $x \otimes y$, in the blocks $(A_0 \oplus E) \otimes (B_0 \oplus F)$. Its $E \otimes F$ block vanishes, so $x_E = 0$ or $y_F = 0$. If $x_E \neq 0$, then $y_F = 0$. The nonzero E-rows have the pairwise distinct directions $v(s+3)$, so only one such row can be nonzero. Its direction determines y, and Lemma 4.4 forces the core block to vanish. The matrix therefore has direction R_s. Transposing gives direction S_t if $y_F \neq 0$. If both complementary parts vanish, Lemma 7.2 gives a Q-direction. Lemma 4.5 and Nakayama's lemma prove reducedness; the support argument in Lemma 7.2 preserves both conclusions under padding.

The only relation among the $r + 1$ matrices is (4.3). Thus an r-subset is independent exactly when it omits one Q_i. Flattening rank r forces every actual decomposition to select such a basis and then uniquely determines its third coefficients. This gives six decompositions. Their immersion and the rank assertions follow from the same proof as Proposition 4.6, for each of the six choices of a core basis.

For the final assertion, form H and the finite unramified incidence near L_r as in Proposition 7.1. The ordered distinct r-tuples, followed by the free permutation quotient, give a finite unramified scheme $F \to H$ with fibre the $r + 1$ unordered r-subsets. Let F_0 be the reduced closure of its independent open subset. That open is the unordered span configuration space; the subset rank criterion makes its span map birational. Hence F_0 is integral and finite birational over this neighbourhood in H. The normalization of H maps finitely and surjectively to F_0:

on affine rings its integral closure contains the finite intermediate domain defining F_0, in the same fraction field. Every point of F_0 thus has a normalization point above it.

The six independent subsets lie in F_0 and have smooth configuration neighbourhoods, so give exactly one smooth normalization point each. To treat a subset omitting $R_s = u_s \otimes v(s+3)$, write $v(s+3) = \sum_{i=1}^{3} \lambda_i r_i$ in the row basis of M. Deform

$$Q_i(z) = (e_i + z\lambda_i u_s) \otimes r_i \quad (1 \le i \le 3), \qquad Q_i(z) = Q_i \quad (4 \le i \le 6),$$

retain every other added matrix, and omit R_s. The sum of the six deformed core representatives is zR_s. For $z \neq 0$ the resulting r matrices are independent: projection of a relation to the core makes its six core coefficients equal, the other added blocks kill all retained added coefficients, and the u_s-row then forces the common coefficient to vanish. Their span extends to L_r at $z = 0$. Indeed, five of the $Q_i(z)$, all retained added matrices, and the fixed R_s form a basis with independent limit; for $z \neq 0$ this basis spans the same space by the displayed sum identity. This is an algebraic arc of spans and configurations showing that the dependent subset belongs to F_0.

For a subset omitting S_t, write $v(t+3) = \sum_{j=1}^{3} \mu_j(-m_j)$ and deform $Q_{3+j}(z) = (-m_j) \otimes (e_j + z\mu_j w_t)$, leaving the other matrices fixed. The sum is zS_t, and the identical column-block argument proves independence and convergence. All $r+1$ distinct subsets therefore lie in F_0, proving the lower bound over L_r. Pulling the normalization back along the rank-r tensor bundle gives the same conclusion over $T_{r;p,q}$. The dependent subsets do not span L_r and cannot be actual decompositions. □

7.3 A proper model for the higher-rank fibres

We retain $W = A \otimes B$, $S = \mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B)$ and the span variety $H \subseteq \mathrm{Gr}(r, W)$. The six matrices Q_i, their five-dimensional span L_0, and $v(z) = (1, z, z^2)^{\mathsf{T}}$ are as above. Put $T_5 = \sum_{i=1}^{5} Q_i \otimes f_i$. In all the formats below,

$$a, b \ge 3, \qquad a + b \ge r + 2, \qquad c \ge r.$$

The unordered span map is birational in this range. Indeed, choose factor lists such that every subset has the largest possible rank. Such lists exist by taking Vandermonde vectors. A linear combination using exactly $s \ge 2$ terms then has matrix rank at least

$$\min(a, s) + \min(b, s) - s \ge 2, \qquad 2 \le s \le r,$$

by (2.3). The r matrices are independent and their span contains precisely their own rank-one directions. Thus a general unordered tuple is uniquely recovered from its span. In characteristic zero this gives birationality, as in the proof of Proposition 2.5.

We shall use a proper model which also records dependent limiting configurations. Let $\mathcal{K}_r$ be the reduced closure of the graph of the span map on independent reduced length-r subschemes of S in

$$\mathrm{Hilb}^r(S) \times H.$$

The projection $\pi : \mathcal{K}_r \to H$ is proper and birational. Properness follows from projectivity of $\mathrm{Hilb}^r(S)$; birationality follows from that of the unordered span map.

Lemma 7.5 (Fibre schemes and the normalization model)**.** *Suppose that $L \in H$ and the scheme $S \cap \mathbb{P}L$ consists of $m \ge r$ distinct reduced points. Then the fibre $\pi^{-1}(L)$ is a reduced finite scheme, each of whose points is an r-element subset of $S \cap \mathbb{P}L$. Every independent such subset occurs. If $\mathcal{K}_r$ is smooth at every point of this fibre, then, after shrinking H about L, the morphism π is its normalization and is unramified, with smooth source. The corresponding conclusions and fibre cardinality hold over every rank-r tensor with flattening image L.*

Proof. The condition that a finite subscheme $Z \subseteq S$ be contained in a varying linear space $\mathbb{P}K$ is closed. It holds on the ordinary span graph and hence on $\mathcal{K}_r$. Therefore $\pi^{-1}(L)$ is a closed subscheme of

$$\mathrm{Hilb}^r(S \cap \mathbb{P}L).$$

For a reduced set of m points this Hilbert scheme is the disjoint union of $\binom{m}{r}$ copies of $\operatorname{Spec}\mathbb{C}$, indexed by its r-element subsets. Every closed subscheme of it is reduced. This proves the fibre assertion, including the absence

of nonreduced collisions and of fibre multiplicities greater than one. Independent subsets belong to the ordinary graph.

Since the fibre is finite, properness permits shrinking the target until π is finite. If the source is smooth along that fibre, remove the proper image of its closed nonsmooth locus as well. The resulting finite birational morphism has normal source and is thus the normalization: its affine coordinate rings are integral over those of the target, have the same fraction field, and are integrally closed. At each point over L the fibre cotangent space is zero. Base change for relative differentials and Nakayama's lemma show that π is unramified there. Removing the finite image of the complement of the unramified locus gives the asserted neighborhood.

Finally, Lemma 2.4 identifies the rank-r flattening locus of $X_r(a,b,c)$ with the bundle of surjections $C^* \to K$ over H. Locally on H this is the product with the open set of full-rank r-by-c matrices. Its pullback to the finite smooth normalization is finite, birational and smooth, hence normal; it is therefore the normalization of the tensor neighborhood. Unramifiedness is preserved by this base change, and the fibre over a fixed surjection is the same reduced fibre as over its image L. This is the tensor transfer in Proposition 7.1. □

Near a configuration of distinct points, ordering the points gives a finite étale cover by an open subset of S^r. Thus smoothness of $\mathcal{K}_r$ may be checked on its ordered graph. The permutation action is free there, including at dependent configurations.

7.4 Diagonal extensions: six points in every rank

Choose complementary coordinate vectors $u_1, \dots, u_t$ in A and $w_1, \dots, w_t$ in B, where $t = r - 5$, and set

$$D_s = u_s \otimes w_s, \qquad L_{\mathrm{diag}} = L_0 \oplus \langle D_1, \dots, D_t \rangle, \qquad T_{\mathrm{diag}} = T_5 + \sum_{s=1}^{t} D_s \otimes f_{5+s}. \tag{7.1}$$

Theorem 7.6 (Diagonal fibres)**.** *For every $r \geq 6$, $a, b \geq r-2$ and $c \geq r$, the tensor T_{diag} has rank and border rank r. The normalization of $X_r(a,b,c)$ has exactly six reduced points over it. All six are smooth and correspond bijectively to its six actual unordered r-term decompositions. There is a Zariski open neighborhood of T_{diag} whose entire normalization is smooth and whose normalization map is unramified. In particular, no boundary normalization point occurs over this tensor. These conclusions hold after any permutation of the three factors.*

The key exclusion uses the following block calculation. It applies to arbitrary rectangular outer blocks, and so includes all the permitted padding of the factors.

Lemma 7.7 (Diagonal block valuation)**.** *Let $R = \mathbb{C}[[z]]$, and let P, N be matrices over R of rank at most three over $\mathbb{C}((z))$, both specializing to M in the upper-left 3-by-3 block and to zero elsewhere. Write*

$$P = \begin{pmatrix} A_P & B_P \\ C_P & E_P \end{pmatrix}, \qquad N = \begin{pmatrix} A_N & B_N \\ C_N & E_N \end{pmatrix}.$$

If $A_P - A_N$, $B_P - B_N$ and $C_P - C_N$ are divisible by z^k, where $k \geq 1$, then $E_P - E_N$ is divisible by z^{k+1}.

Proof. The A blocks are invertible over R. The rank bounds give the exact identities

$$E_P = C_P A_P^{-1} B_P, \qquad E_N = C_N A_N^{-1} B_N.$$

Both cross blocks of each matrix are divisible by z. Moreover,

$$\begin{aligned} E_P - E_N &= C_P A_P^{-1}(B_P - B_N) + (C_P - C_N) A_N^{-1} B_N \\ &\quad + C_P (A_P^{-1} - A_N^{-1}) B_N, \\ A_P^{-1} - A_N^{-1} &= -A_P^{-1}(A_P - A_N) A_N^{-1}. \end{aligned}$$

The first two summands are divisible by z^{k+1} and the last by z^{k+2}, proving the claim. □

Proof of Theorem 7.6. A matrix in L_{diag} is block diagonal, with one core 3-by-3 block and t scalar diagonal blocks. The rank of such a matrix is the sum of the ranks of its nonzero blocks. A nonzero rank-one matrix consequently has only one nonzero block. By Lemma 7.2, the projective rank-one intersection is exactly

$$S \cap \mathbb{P}L_{\mathrm{diag}} = \{[Q_1], \dots, [Q_6], [D_1], \dots, [D_t]\}. \tag{7.2}$$

This equality holds scheme-theoretically with all points reduced. Indeed, at a core point choose a nonzero core pivot and normalize both factor coordinates at that pivot to one. The off-block entries in its row and column force every complementary factor coordinate to vanish scheme-theoretically. The remaining intersection is the reduced core intersection. At $[D_s]$ use its diagonal entry as pivot: the vanishing of every other entry in that pivot row and column forces all other factor coordinates to vanish. The same argument eliminates unused coordinates under arbitrary padding.

The only relation among the $r+1$ displayed generators is $\sum_{i=1}^{6} Q_i = 0$. Thus their independent r-subsets are exactly the six subsets omitting one core point. Each is a basis of L_{diag}. The third flattening of T_{diag} has rank r and image L_{diag}. An r-term decomposition must give a rank-one basis of that image, and conversely each such basis determines the third factors uniquely. It therefore gives exactly one unordered decomposition into nonzero pure summands. There are exactly six such decompositions, and the displayed sum and the flattening lower bound give rank and border rank r.

We must exclude the remaining r-subsets from the proper graph. By Lemma 7.5, they are the only possible boundary configurations. Fix one omitting D_s; it contains all six core points and every other D_j. Suppose it occurs in $\mathcal{K}_r$. Choose a curve through this point with generic point in the ordinary independent span graph, normalize the curve and complete its local ring. After lifting to the ordering cover, this gives rank-one matrix arcs over $R = \mathbb{C}[[z]]$,

$$p_i(z) \longrightarrow Q_i \quad (1 \le i \le 6), \qquad d_j(z) \longrightarrow D_j \quad (j \ne s),$$

whose generic span has dimension r and specializes to L_{diag}. Representatives regular in their pivot charts may be chosen with exactly the indicated limits.

The first-factor vectors of the retained $d_j(z)$ are independent over R, as are their second-factor vectors. Completing these lists to bases gives changes of basis in A and B, congruent to the identity modulo z, which make all retained $d_j(z)$ equal to D_j. These changes preserve the limiting subspace. The moving span is a rank-r subbundle of the constant W-bundle. Choose a regular section $Z(z)$ of it satisfying $Z(0) = D_s$, and write over $\mathbb{C}((z))$

$$Z(z) = \sum_{i=1}^{6} \alpha_i(z) p_i(z) + \sum_{j \ne s} \beta_j(z) D_j.$$

Some coefficient has a pole, since the limiting tuple does not span D_s. Let $k \ge 1$ be the largest pole order. After multiplication by z^k, reduction modulo z gives a nonzero relation on the limiting tuple. The six core coefficients therefore all have the same nonzero constant λ, and every retained diagonal coefficient has constant zero. Dividing by λ, denote the integral coefficients by $a_i(z)$ and $b_j(z)$; then

$$a_i(0) = 1, \qquad b_j(0) = 0.$$

Set

$$P = \sum_{i=1}^{3} a_i p_i, \qquad N = -\sum_{i=4}^{6} a_i p_i.$$

Both matrices have rank at most three and specialize to M in the core block. Their relation is

$$P - N + \sum_{j \ne s} b_j D_j = \lambda^{-1} z^k Z(z). \tag{7.3}$$

Its core and cross blocks show that the three differences in Lemma 7.7 are divisible by z^k. The whole lower-right block of $P - N$ is consequently divisible by z^{k+1}. At the (u_s, w_s) coordinate, all retained D_j vanish, whereas the right side of (7.3) has valuation exactly k, because $Z(0) = D_s$. This is a contradiction.

Thus the proper fibre contains only the six independent subsets. At each, the graph is an open chart in the smooth unordered configuration space. Lemma 7.5 now proves the normalization assertions, including the multiplicity one of every fibre point and the tensor-neighborhood statement. The dimension assumptions give $a+b \ge 2r-4 \ge r+2$, as required for generic birationality. Permuting tensor factors gives the final assertion. □

7.5 The seventh point at rank six

Theorem 7.8 (The rank-six row-extension fibre)**.** *Let $a \ge 4$, $b \ge 3$, $a+b \ge 8$ and $c \ge 6$. For a new coordinate vector $u \in A$ outside A_0, put*

$$R = u \otimes v(4), \qquad L = L_0 \oplus \mathbb{C}R, \qquad T_{\mathrm{row},6} = T_5 + R \otimes f_6.$$

The tensor has rank and border rank six. The normalization of $X_6(a,b,c)$ has exactly seven reduced points over it, all smooth. Six correspond to actual six-term decompositions. The seventh corresponds to the dependent configuration of all six core directions $[Q_1],\dots,[Q_6]$ and is not an actual decomposition. On a Zariski open neighborhood of this tensor the entire normalization is smooth and its morphism is unramified. Exchanging A and B gives the column-extension statement with $a \geq 3$, $b \geq 4$, $a+b \geq 8$ and $c \geq 6$; all factor permutations are allowed.

Proof. Proposition 7.4 gives the reduced intersection

$$S \cap \mathbb{P}L = \{[Q_1],\dots,[Q_6],[R]\},$$

the six actual decompositions and the rank assertions. The six subsets omitting a core point are independent. By Lemma 7.5, the only remaining candidate in the proper fibre is the six-core configuration. We calculate the graph scheme at that configuration.

Let $\mathcal{C}_6 \subseteq S^6$ be the ordered distinct-point configuration space, of dimension $n = 6(a+b-2)$, and let $x = ([Q_1],\dots,[Q_6])$. Choose regular nonzero representatives P_i near x and a fixed invertible five-row coordinate minor of the first five columns of $(P_1,\dots,P_6)$. Subtract from P_6 the unique linear combination of the first five columns which agrees on those pivot rows, and call the residual vector q. Its remaining coordinates generate the ideal I of all six-by-six minors. This is an equality of ideals, since row reduction using the invertible pivot changes the column matrix to an identity block with final residual column q. Write $D = V(I)$ for this exact determinantal subscheme.

Equivalently, impose the relation

$$\sum_{i=1}^{6} \lambda_i P_i = 0, \qquad \lambda_6 = 1. \tag{7.4}$$

The pivot solves the other five coefficients uniquely and regularly, so eliminating them identifies this incidence scheme with D. All six coefficients at x are nonzero; we may restrict to the locus where they are units. Lift the projective points to nonzero factor vectors a_i, b_i. This is a smooth torus torsor of relative dimension twelve. Absorbing λ_i into a_i turns (7.4) into

$$\mathsf{A}\mathsf{B}^{\mathsf{T}} = 0, \qquad \mathsf{A} = (a_1,\dots,a_6), \quad \mathsf{B} = (b_1,\dots,b_6), \tag{7.5}$$

with the five relation coordinates otherwise free.

Here is an explicit scheme chart for (7.5). Write

$$\mathsf{A} = [A_1\ A_2], \qquad \mathsf{B}^{\mathsf{T}} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 = [D_0\ E_0], \quad B_2 = [C_0\ F_0],$$

where C_0, D_0 are three-by-three. At the core, A_1 has a fixed invertible three-row minor and C_0 is invertible. On that open chart the product equations are exactly

$$A_1 D_0 + A_2 C_0 = 0, \qquad A_1 E_0 + A_2 F_0 = 0.$$

They are scheme-theoretically equivalent to

$$A_2 = -A_1 D_0 C_0^{-1}, \qquad E_0 = D_0 C_0^{-1} F_0. \tag{7.6}$$

Indeed, the first equation solves A_2; after substitution, a left inverse for A_1 turns the second into the displayed equation for E_0. There are no further equations. The free coordinates A_1, C_0, D_0, F_0 have total dimension

$$3a + 9 + 9 + 3(b-3) = 3a + 3b + 9.$$

Adding the five relation coordinates and descending through the 12-dimensional smooth torsor shows that D is smooth of dimension $3a+3b+2$. Its codimension in $\mathcal{C}_6$ is

$$h = 3a + 3b - 14. \tag{7.7}$$

In particular the exact maximal-minor ideal, and not merely its radical, is the ideal of a smooth center.

At x, differentiating the residual column gives, up to a common nonzero scalar,

$$dq_x(\delta P_1, \dots, \delta P_6) = \sum_{i=1}^{6} \delta P_i \pmod{L_0}.$$

The sum of the affine rank-one tangent spaces at the Q_i is

$$U = A_0 \otimes B + A \otimes B_0, \qquad \dim U = 3a + 3b - 9.$$

To see the equality, use the tangent formula $T_{a_i \otimes b_i} \widehat{S} = a_i \otimes B + A \otimes b_i$, and the facts that the a_i span A_0 and the b_i span B_0. Projective representative changes discard only radial directions, which lie in L_0. Thus the differential image is exactly U/L_0, of dimension h. Its kernel is $T_x D$, and it induces an isomorphism

$$N_{D/\mathcal{C}_6,x} \simeq U/L_0. \tag{7.8}$$

The Plücker coordinates of the rational span map are

$$P_1 \wedge \cdots \wedge P_5 \wedge P_6 = P_1 \wedge \cdots \wedge P_5 \wedge q.$$

Their base ideal is exactly I. The closure of this rational graph is therefore the blowup $\mathrm{Bl}_D(\mathcal{C}_6)$: on affine charts the Rees algebra $\bigoplus_{j \geq 0} I^j$ embeds in the polynomial algebra with its degree parameter, and its Proj is the closure of the graph where I is nonzero. The Plücker relations hold on this closure, so using the Grassmannian as target gives the same graph. Both $\mathcal{C}_6$ and D are smooth, hence this blowup is smooth. Explicitly, in étale local coordinates with $I = (x_1, \dots, x_h)$, the chart indexed by x_j has equations $x_i = x_j z_i$ for $i \neq j$ and is smooth after eliminating those x_i.

The exceptional fibre over x is $\mathbb{P}(N_{D/\mathcal{C}_6,x})$. Under (7.8), its map to the Grassmannian is

$$[w] \longmapsto L_0 + \mathbb{C}w, \qquad [w] \in \mathbb{P}(U/L_0).$$

Different lines give different six-dimensional subspaces. Since $R \in A \otimes B_0 \subseteq U$ and $R \notin L_0$, exactly one exceptional point maps to $L = L_0 + \mathbb{C}R$. It is smooth. The free permutation quotient preserves this local smoothness and gives exactly one point over the unordered six-core configuration.

The complete proper fibre consequently consists of this point and the six smooth ordinary points. By Lemma 7.5, all seven have multiplicity one, and they give the entire smooth, unramified normalization over a neighborhood of L and then over a neighborhood of $T_{\mathrm{row},6}$. The exceptional configuration spans only L_0, so it cannot represent an actual six-term decomposition of a tensor whose third flattening has rank six. The symmetry statements follow by permuting the factors. □

7.6 All ranks with a three-dimensional matrix factor

Theorem 7.9 (The three-dimensional-factor row family)**.** *Let $r \geq 6$, $a \geq r-1$ and $c \geq r$, and let $B = B_0$ have dimension exactly three. For independent extra coordinate vectors $u_1, \dots, u_{r-5}$ in A, set*

$$R_s = u_s \otimes v(s+3), \qquad L_r = L_0 \oplus \langle R_1, \dots, R_{r-5} \rangle, \qquad T_{\mathrm{row}} = T_5 + \sum_{s=1}^{r-5} R_s \otimes f_{5+s}. \tag{7.9}$$

The tensor has rank and border rank r. The normalization of $X_r(a,3,c)$ has exactly $r+1$ reduced points over it, all smooth. Six correspond to actual r-term decompositions. The other $r-5$ correspond to the dependent configurations which omit one R_s and contain all six Q_i. There is a Zariski open neighborhood of T_{row} whose entire normalization is smooth and whose normalization morphism is unramified. All factor permutations of this statement hold.

Proof. By Proposition 7.4, the rank-one intersection is exactly the $r+1$ listed reduced points, and the only relation among them is the six-core relation. There are six independent r-subsets, giving precisely the six actual decompositions; the rank and border rank are r. The other $r-5$ subsets each omit one added point. Lemma 7.5 reduces exhaustion and smoothness to the graph at these dependent configurations.

Fix one such ordered configuration $x = (p_1, \dots, p_r)$ and reorder it so that the first $r-1$ points are independent. Write K for their $(r-1)$-dimensional span at x. In local projective charts choose nonzero affine representatives

$P_i(y)$ and invert a fixed $(r-1)$-row minor of the first $r-1$ columns. Their spans form a subbundle $K(y)$ of the constant W-bundle, whose quotient is locally free of rank

$$h = \dim W - (r-1) = 3a - r + 1.$$

In a quotient trivialization let $q(y)$ be the residual image of $P_r(y)$. Elementary row operations using the pivot show that the coordinates of q generate the ideal of all r-by-r minors. Thus the rank-deficiency subscheme is exactly $D = \{q = 0\}$, with this ideal, not just this zero set.

In the unique relation $\sum_i \lambda_i P_i(x) = 0$, the six core coefficients are nonzero and those of the retained added points are zero. Differentiating the residual column yields, up to a common nonzero scalar and quotient coordinates,

$$dq_x(\delta P_1, \dots, \delta P_r) = \sum_{i\,\mathrm{core}} \lambda_i \delta P_i \pmod K. \tag{7.10}$$

The three row-core matrices are $Q_i = e_i \otimes r_i$, where the rows r_1, r_2, r_3 of the invertible matrix M form a basis of B. Their affine tangent spaces contain $A \otimes r_i$; their sum therefore contains all of $A \otimes B = W$. Each relevant coefficient in (7.10) is nonzero. Passing to projective tangent spaces removes only the radial directions Q_i, which already lie in K. Consequently dq_x is surjective onto W/K.

It follows from the Jacobian criterion that q is smooth near x and that D is smooth of codimension h. Equivalently, the coordinates $q_1, \dots, q_h$ form part of an étale local coordinate system. As in the rank-six calculation, the local ordered span graph is the blowup of the exact maximal-minor ideal $(q_1, \dots, q_h)$. In its q_j chart the equations are

$$q_i = q_j z_i \quad (i \neq j).$$

Eliminating these q_i leaves q_j, the z_i, and the complementary smooth coordinates free. Thus the graph is smooth at every point above x.

The exceptional fibre over x is $\mathbb{P}(W/K)$. Its map to the Grassmannian is the usual isomorphism from $\mathbb{P}(W/K)$ to the variety of r-dimensional subspaces containing K,

$$[w] \longmapsto K + \mathbb{C}w.$$

Since $K \subset L_r$, the prescribed target L_r selects exactly one exceptional point. In particular that point exists in the graph, is smooth, and is the only point over this dependent configuration and L_r. The ordering cover is étale, so the same is true in the unordered graph.

This gives one point for each of the $r-5$ dependent subsets and one ordinary smooth point for each of the six independent subsets. Lemma 7.5 proves that this is the entire reduced fibre, that it has $r+1$ points, and that the normalization is smooth and unramified on a neighborhood. Its tensor transfer proves the assertions over T_{row}. Here $a + 3 \geq r + 2$ supplies the required generic birationality. A dependent subset spans only an $(r-1)$-space and hence cannot give an actual decomposition of this rank-r flattening. □

The equality $B = B_0$ is a hypothesis of Theorem 7.9: it is precisely what makes the three row-core tangent spaces cover $A \otimes B$ in the residual derivative. The first factor and the third factor may be enlarged throughout the stated ranges. Theorem 7.8 separately permits enlargement of both matrix factors at rank six. The three theorems above do not address the mixed row/column family in ranks $r \geq 7$ with both matrix factors enlarged; specified such families will be treated below. Their neighborhood statements imply no global smoothness, Cohen–Macaulayness or rational singularities of a normalization.

7.7 The completed local ring of the rank-five circuit

The six normalization points at the rank-five circuit admit an exact description of their gluing. We retain the matrices $M, Q_1, \dots, Q_6$ of (4.1) and (4.2), their span $L_0 \subset A_0 \otimes B_0$ with $\dim A_0 = \dim B_0 = 3$, and the tensor $T_5 = \sum_{i=1}^{5} Q_i \otimes f_i$ of (4.5). The subspaces and tensor are embedded by coordinate inclusions whenever the ambient factor dimensions are larger.

Theorem 7.10. *Let $a, b \geq 3$, $a + b \geq 7$ and $c \geq 5$, and put $X = X_5(a, b, c)$. At the specified padded tensor $T = T_5$, set*

$$q = 2(a + b - 6), \qquad h = 3a + 3b + 5c + 2. \tag{7.11}$$

There is an isomorphism of complete local $\mathbb{C}$-algebras

$$\widehat{\mathcal{O}}_{X,T} \simeq \frac{\mathbb{C}[[z_1, \dots, z_h,\ x_{i,j} : 1 \le i \le 6,\ 1 \le j \le q]]}{(x_{i,j}x_{k,\ell} : i \neq k)}. \tag{7.12}$$

In particular,

$$\dim \mathcal{O}_{X,T} = h + q = 5(a + b + c - 2), \tag{7.13}$$

$$\operatorname{depth} \mathcal{O}_{X,T} = h + 1 = 3a + 3b + 5c + 3, \tag{7.14}$$

$$\operatorname{edim} \mathcal{O}_{X,T} = h + 6q, \qquad e(\mathcal{O}_{X,T}) = 6. \tag{7.15}$$

Here e denotes Hilbert–Samuel multiplicity. The normalization of the completed ring is

$$\prod_{i=1}^{6} \mathbb{C}[[z_1, \dots, z_h, x_{i,1}, \dots, x_{i,q}]]. \tag{7.16}$$

Every two branches of the completed local ring meet the same reduced smooth h-dimensional formal germ. The conductor into (7.16) is generated by all the $x_{i,j}$, and its quotient is $\mathbb{C}[[z_1, \dots, z_h]]$.

The assertions are preserved by simultaneously permuting the tensor factors and transporting this specified witness.

Proof. The core chart. By Lemma 7.2, the scheme

$$\mathbb{P}L_0 \cap \operatorname{Seg}(\mathbb{P}A_0 \times \mathbb{P}B_0)$$

consists of exactly the six reduced points $[Q_i]$. Every five of the Q_i are independent, and their only relation, up to scalar, is $\sum_i Q_i = 0$. Write $W_0 = A_0 \otimes B_0$ and consider

$$G_0 = \operatorname{Gr}(5, W_0), \qquad Z_0 = \{(L, p) \in G_0 \times \operatorname{Seg}(\mathbb{P}A_0 \times \mathbb{P}B_0) : p \in \mathbb{P}L\}.$$

The Grassmannian G_0 is smooth of dimension 20. Projection of Z_0 to the Segre is a Grassmann bundle: over a line $p \subset W_0$ its fiber is $\operatorname{Gr}(4, W_0/p)$, of dimension 16. Thus Z_0 is also smooth of dimension 20.

The projection $Z_0 \to G_0$ is proper, and its fiber at L_0 is the six-point reduced scheme just described. Upper semicontinuity of fiber dimension, followed by proper quasi-finiteness, makes the projection finite after shrinking around L_0. At each point of that fiber the relative cotangent space is zero, so the morphism is unramified there. The source and target are smooth of the same dimension; its differential is therefore an isomorphism, and the morphism is étale at those points. Remove the proper image of the non-étale locus. After a further shrinking it is finite étale of degree six.

After an étale base change we may label the six sections $p_1, \dots, p_6$. Work on a sufficiently small smooth étale neighborhood U_0 of L_0, with a chosen complex point over L_0. Trivialize the line bundles of the six sections and choose regular nonzero representatives

$$q_i = a_i \otimes b_i \in L, \qquad \sum_{i=1}^{6} q_i = 0. \tag{7.17}$$

To obtain the last equality, start with arbitrary representatives. Every five remain a basis near L_0, so the unique relation among the six has all coefficients invertible. Absorb those coefficients into the representatives and into one of their factors. Choosing a nonzero matrix entry of each rank-one representative makes its factorization into a_i, b_i regular. After shrinking again, the a_i span A_0 and the b_i span B_0. The completed local ring of U_0 at the chosen complex point is the same as that of G_0 at L_0. Only this core incidence has been made étale; no flatness of the larger-format incidence is being assumed.

Graph coordinates for nearby five-planes. Choose complements

$$A = A_0 \oplus U, \qquad B = B_0 \oplus V, \qquad \alpha = \dim U = a - 3, \quad \beta = \dim V = b - 3.$$

Near L_0, projection to W_0 identifies a five-plane with the graph over its projected $L \in G_0$ of three maps

$$F : L \longrightarrow U \otimes B_0, \qquad G : L \longrightarrow A_0 \otimes V, \qquad K : L \longrightarrow U \otimes V. \tag{7.18}$$

Pull back this open Grassmannian chart to U_0. Projection is injective on each such graph, so a nonzero rank-one point of the graph projects to a nonzero rank-one point of L. Its projective image is one of the six p_i, and there is at most one lifted line over each p_i.

The lift whose core component is $q_i = a_i \otimes b_i$ is rank one exactly when there are $u_i \in U$, $v_i \in V$ with

$$F(q_i) = u_i \otimes b_i, \qquad G(q_i) = a_i \otimes v_i, \qquad K(q_i) = u_i \otimes v_i. \tag{7.19}$$

Indeed, expanding $(a_i + u_i) \otimes (b_i + v_i)$ gives these three components. Conversely, a rank-one tensor with core component exactly $a_i \otimes b_i$ has this factorization after reciprocal rescaling of its two factors. This rescaling does not change the equations.

Residues and their common zero locus. Consider the vector-bundle map

$$\begin{aligned} \Phi_A : \operatorname{Hom}(L, U \otimes B_0) &\longrightarrow \bigoplus_{i=1}^{6} U \otimes (B_0/\langle b_i \rangle), \\ F &\longmapsto \big(F(q_i) \bmod U \otimes \langle b_i \rangle\big)_{i=1}^{6}. \end{aligned} \tag{7.20}$$

This map is surjective. Given arbitrary quotient classes, choose representatives $w_i \in U \otimes B_0$. Since the b_i span B_0, there exist $u_i \in U$ such that

$$\sum_i u_i \otimes b_i = -\sum_i w_i.$$

The six vectors $w_i + u_i \otimes b_i$ then sum to zero. By (7.17) and the independence of every five q_i, they are the values of a unique linear map on L. This also proves surjectivity when $\alpha = 0$.

The domain of Φ_A has rank 15α and its target has rank 12α, so its kernel has rank 3α. It consists precisely of

$$F = (P \otimes \mathrm{id}_{B_0})|_L, \qquad P \in \operatorname{Hom}(A_0, U). \tag{7.21}$$

These maps lie in the kernel. The map $P \mapsto F$ is injective because the a_i span A_0; equality with the kernel follows from the ranks. The same construction for G gives a surjection

$$\Phi_B : \operatorname{Hom}(L, A_0 \otimes V) \longrightarrow \bigoplus_{i=1}^{6} (A_0/\langle a_i \rangle) \otimes V$$

whose kernel is exactly

$$G = (\mathrm{id}_{A_0} \otimes Q)|_L, \qquad Q \in \operatorname{Hom}(B_0, V). \tag{7.22}$$

After shrinking U_0, split these surjections of vector bundles and choose frames for their rank-two quotient factors. This gives coordinates P, Q and six groups

$$x_i = (x_{i,1}, \dots, x_{i,q}), \qquad q = 2\alpha + 2\beta.$$

The first two equations of (7.19) hold for every index other than i if and only if $x_k = 0$ for every $k \neq i$. If every residue vanishes, they hold for all six indices with $u_i = P(a_i)$ and $v_i = Q(b_i)$. Applying $P \otimes Q$ to (7.17) gives $\sum_i u_i \otimes v_i = 0$, so the six last equations are consistent and determine the common value

$$H_0 = (P \otimes Q)|_L. \tag{7.23}$$

This common locus is smooth of dimension $20+3\alpha+3\beta$. The assertion follows from these exact family coordinates, rather than only from tangent dimensions.

Simultaneous straightening of the six configuration branches. Fix i. On the residue axis $x_k = 0$ for $k \neq i$, require the five points with indices $k \neq i$ to lift. Their q_k form a basis of L. The first two equations of (7.19) determine u_k, v_k uniquely, and the last equations determine K on that basis. Denote this map by

$$K = H_i(L, P, Q, x_i).$$

It is regular: the basis determinant and the selected nonzero factor entries that must be inverted are units on the chosen neighborhood. It is bilinear in the F, G variables over the core chart. At $x_i = 0$ it agrees with H_0. Thus the locus B_i containing these five lifted lines is an embedded smooth closed graph with coordinates L, P, Q, x_i.

Extend each H_i to all residue coordinates by ignoring x_k for $k \neq i$, and make the single change of coordinates

$$H = H_0 + \sum_{i=1}^{6} (H_i - H_0), \qquad K' = K - H. \tag{7.24}$$

This is invertible because H depends only on the other graph coordinates. On the ith residue axis every H_k with $k \neq i$ equals H_0, so $H = H_i$. Consequently all six graphs become

$$B_i = \{K' = 0,\ x_k = 0 \text{ for every } k \neq i\}. \tag{7.25}$$

This simultaneous description proves independence of all the normal groups and their scheme-theoretic pair intersections. Pairwise transversality alone is not needed to infer either conclusion.

Exhaustion of limiting branches. Let $S = \mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B)$, and let $\mathcal{H}$ be the reduced closure in $\mathrm{Gr}(5, A \otimes B)$ of the spans of five independent points of S. We next show that its reduced local germ is exactly $\bigcup_{i=1}^{6} B_i$.

The point incidence

$$Z = \{(L, p) : p \in \mathbb{P}L \cap S\} \longrightarrow \mathrm{Gr}(5, A \otimes B)$$

is proper. Its fiber at the padded L_0 is still exactly the six reduced points, by Lemma 7.2: padding adds neither rank-one directions nor tangent directions in that fiber. As for the core incidence, proper quasi-finiteness makes Z finite on an ambient neighborhood of L_0. The zero relative cotangent spaces at the six reduced fiber points give unramifiedness there. Remove the closed image of the complement of the unramified locus to obtain a finite unramified morphism over the whole chosen neighborhood. This does not assert that this larger-format morphism is flat.

In the fivefold fiber product of Z over that neighborhood, let D be the locus of ordered pairwise distinct points. The diagonal of a separated unramified morphism is open and closed. Each coincidence locus is therefore open and closed, and so is their complement D. It follows that D is finite over the ambient neighborhood. Every tuple in its fiber over L_0 is independent. Dependence is closed on D; removing its closed image makes every tuple in D independent after another shrinking.

The image of D is closed and is now exactly the locus of five-planes spanned by five distinct Segre points. It is contained in $\mathcal{H}$ and contains its dense defining span locus restricted to this neighborhood. Since closure commutes with restriction to an open set, it equals $\mathcal{H}$ there. On the graph chart all possible rank-one points lift the six labeled core points, and every choice of five has independent core projections. Thus this image is precisely $\bigcup_i B_i$. The étale pullback of the reduced variety $\mathcal{H}$ is reduced, and every B_i is smooth and reduced. Equality of these closed sets is therefore equality of the reduced schemes. In particular, coincident or dependent limiting configurations create no additional branch.

Eliminating K' in (7.25), the local span germ is the union of six coordinate q-planes over the smooth base (L, P, Q) of dimension $20 + 3\alpha + 3\beta$. Its ideal is generated by all products of coordinates from different residue groups: a monomial belongs to the intersection of the six branch ideals exactly when its support meets at least two groups. The same statement holds after completion, either by this monomial description or by flatness of completion applied to the finite intersection of the branch ideals.

From five-planes to tensors. The flattening $C^* \to A \otimes B$ of T has rank five and image L_0. On the open rank-five locus, X is the bundle of surjections $C^* \twoheadrightarrow L$ over $\mathcal{H}$. Indeed, for a span of five independent rank-one matrices, every surjection onto that span gives a five-term tensor decomposition. Conversely, a rank-five tensor with a five-term decomposition has independent first-two-factor matrices and independent third factors. Such tensors form a dense subset. Taking closures commutes with the smooth open bundle projection, so the reduced rank-five locus is exactly the indicated bundle.

Its fiber is an open subset of $\mathrm{Hom}(C^*, L)$, of dimension $5c$, and it is locally trivial near T. It therefore adds $5c$ formal parameters to the span germ. The 20 smooth core parameters, the $3\alpha + 3\beta$ kernel parameters and these $5c$ tensor parameters give

$$h = 20 + 3\alpha + 3\beta + 5c = 3a + 3b + 5c + 2.$$

The étale base change preserves the completed local ring at the chosen complex point. Together with the span-germ calculation this proves (7.12).

Depth and numerical invariants. Put

$$R_q = \mathbb{C}[[x_{i,j} : 1 \le i \le 6,\ 1 \le j \le q]]/(x_{i,j}x_{k,\ell} : i \ne k), \qquad B_q = \prod_{i=1}^{6} \mathbb{C}[[x_{i,1}, \dots, x_{i,q}]].$$

Each element of R_q has a unique expression as a constant plus six series, each of zero constant term and supported in a single residue group. Hence R_q embeds in B_q as the tuples with a common constant term, and there is an exact sequence of R_q-modules

$$0 \longrightarrow R_q \longrightarrow B_q \longrightarrow \mathbb{C}^5 \longrightarrow 0. \tag{7.26}$$

The last map subtracts the sixth constant term from each of the first five. The ring R_q is reduced of dimension q. Every branch factor of B_q is a regular q-dimensional quotient of R_q, so its depth as an R_q-module is q. Depth of a finite direct sum is the minimum of the depths of the summands, giving $\operatorname{depth}_{R_q} B_q = q$. The nonzero finite-length module $\mathbb{C}^5$ has depth zero. Our range gives $q = 2(a+b-6) \ge 2$; the depth lemma applied to (7.26) therefore gives

$$\operatorname{depth} R_q = 1.$$

Multiplication by a formal variable is injective over every coefficient ring, and successive quotients remove one variable. Thus $z_1, \dots, z_h$ is a regular sequence in $R_q[[z_1, \dots, z_h]]$. Its depth is $h+1$ and its dimension is $h+q$. Dimension and depth are preserved by completion of a Noetherian local ring, proving (7.13) and (7.14). The defining equations in (7.12) are quadratic, so the embedding dimension is $h+6q$. The associated graded Hilbert series is

$$(1-t)^{-h}\big(6(1-t)^{-q} - 5\big). \tag{7.27}$$

Multiplication by $(1-t)^{h+q}$ and evaluation at $t=1$ gives multiplicity six. Embedding dimension and Hilbert–Samuel multiplicity also agree with those of the original local ring, which proves (7.15).

Normalization, conductor and the common intersection. The normalization of the completed ring is the product of its six regular branch rings in (7.16). In that product, the original ring consists of tuples having the same restriction to $\mathbb{C}[[z_1, \dots, z_h]]$. If such a tuple belongs to the conductor, multiplying by the idempotent of any one branch must again yield a tuple with common restriction. The other five restrictions are zero, so the original common restriction must be zero. Conversely, a tuple whose six restrictions are zero remains of common zero restriction after multiplication by any element of the product. It therefore belongs to the conductor.

This identifies the conductor with the ideal generated by all $x_{i,j}$. Its quotient is $\mathbb{C}[[z_1, \dots, z_h]]$, so the ideal is radical. The sum of any two distinct branch ideals also kills all the x variables. Every pair of branches therefore meets in exactly this same reduced smooth h-dimensional formal germ. Finally, a simultaneous permutation of the tensor factors is a linear isomorphism of the ambient tensor spaces and carries the specified witness and its local ring to the transported ones. □

Remark 7.11. For the padded $6 \times 6 \times 6$ witness, $h = 68$ and $q = 12$, so the dimension is 80 and the depth is 69. For the $3 \times 4 \times 5$ witness, $h = 48$ and $q = 2$, so the dimension is 50 and the depth is 49. In general the difference between dimension and depth is $q - 1$. The common branch locus has codimension q in this local model and supports its S_2 defect. These statements concern the explicit rank-five witnesses. They give no formula for the depth at the homogeneous vertex, no global normalization description, and no higher-rank fiber formula. In particular, depth 69 at the $6 \times 6 \times 6$ witness is not a claim of depth less than two there.

7.8 Boundary models for one-sided and mixed extensions

We retain the circuit matrices $Q_1, \dots, Q_6$, their span L_0, and $v(s) = (1, s, s^2)^{\mathsf{T}}$ from (4.2) and the row–column construction. Thus $\sum_{i=1}^{6} Q_i = 0$, and A_0, B_0 denote the old three-dimensional coordinate subspaces. All graph and image closures below have their reduced structures. Normalization is taken in the variety's own function field.

For nonnegative integers s, m, k, α, β, define the subalgebras

$$\mathcal{R}_{s,m} = \mathbb{C}\left[t, \mu_j, y_{ij}, \frac{\mu_j y_{ij}}{t} : 1 \le j \le s,\ 1 \le i \le m\right], \tag{7.28}$$

$$\mathcal{B}_{k,\alpha,\beta} = \mathbb{C}\left[t, p, q_j, x_j, z_j, u_h, v_{ij}, \frac{pz_j + q_j x_j}{t}, \frac{pu_h}{t}, \frac{q_j v_{ij}}{t} : 1 \le j \le k,\ 1 \le h \le \beta,\ 1 \le i \le \alpha\right]. \tag{7.29}$$

Each is the displayed subalgebra of the Laurent polynomial ring in its algebraically independent base variables with t inverted. Empty index sets are allowed. In $\mathcal{B}$, α counts the v-rows and β the u-variables. Give every base variable degree one; the displayed fractions also have degree one. The *homogeneous origin* is the point where all these generators vanish.

We use *smooth equivalence* in the following precise sense: a smooth local cover of the indicated germ is étale over the product of the model germ with a smooth germ. The covers include the choices of factor vectors and the ordering of configurations. No canonical coordinate isomorphism on the unframed normalization is asserted. The *tangent excess* of a local ring is its embedding dimension minus its Krull dimension.

Theorem 7.12 (One-sided extensions, D)**.** *Let*

$$r \ge 6, \qquad a \ge r-2, \qquad b \ge 3, \qquad c \ge r, \qquad a+b \ge r+2.$$

In $A = \mathbb{C}^a$, choose distinct new coordinate vectors $u_1, \dots, u_{r-5}$ complementary to A_0, and put

$$\begin{aligned} R_s &= u_s \otimes v(s+3) \quad (1 \le s \le r-5), \\ L_r &= L_0 \oplus \langle R_1, \dots, R_{r-5} \rangle, \\ T_{\mathrm{row}} &= \sum_{i=1}^{5} Q_i \otimes f_i + \sum_{s=1}^{r-5} R_s \otimes f_{5+s}. \end{aligned} \tag{7.30}$$

Here $v(s+3) \in B_0 \subset B = \mathbb{C}^b$, even when $b > 3$, and $f_1, \dots, f_r$ are coordinate vectors in $\mathbb{C}^c$. The ordinary rank and border rank of T_{row} equal r. Its entire normalization fibre in $X_r(a,b,c)$ consists of exactly $r+1$ reduced points. Six correspond to the precisely six actual unordered r-term decompositions and are smooth. The other $r-5$ points correspond to the dependent configurations containing all six Q-directions and all R-directions except one; they are not actual r-term decompositions.

Put $\ell = r-6$ and $m = b-3$. Every boundary point is smoothly equivalent to the homogeneous origin of $\operatorname{Spec} \mathcal{R}_{\ell,m}$. This ring is normal, with tangent excess ℓm; its origin is smooth precisely when $\ell m = 0$. Thus all boundary points are smooth precisely when $r = 6$ or $b = 3$, and all are singular when $r \ge 7$ and $b \ge 4$. Every such point has dimension $r(a+b+c-2)$ and embedding dimension

$$r(a+b+c-2) + (r-6)(b-3).$$

There is a Zariski open neighbourhood of T_{row} over which normalization is unramified. If $r = 6$ or $b = 3$, its entire source over a sufficiently small such neighbourhood is smooth. All conclusions transport under permutation of the tensor factors together with the witness; in particular, transposition gives the added-column version.

The cases $r = 6$ and $b = 3$ recover the corresponding one-sided families above. The disjoint diagonal witnesses are different. Theorem 7.12 does not apply to simultaneous row and column additions and makes no global smoothness, Cohen–Macaulayness, rational-singularity or exact-depth assertion about the normalization.

Theorem 7.13 (One row and several columns, E1)**.** *Let*

$$r \ge 7, \qquad a \ge 4, \qquad b \ge r-3, \qquad c \ge r, \qquad a+b \ge r+2, \qquad k = r-6.$$

Choose a new A coordinate u and new B coordinates $w_1, \dots, w_k$, and set

$$\begin{aligned} R &= u \otimes v(4), \qquad S_j = v(j+3) \otimes w_j, \\ L &= L_0 \oplus \mathbb{C}R \oplus \langle S_1, \dots, S_k \rangle, \\ T &= \sum_{i=1}^{5} Q_i \otimes f_i + R \otimes f_6 + \sum_{j=1}^{k} S_j \otimes f_{6+j}. \end{aligned} \tag{7.31}$$

This is $T_{r;4,r-3}$, of support dimensions $(4, r-3, r)$, padded to the stated format. Its rank and border rank equal r, and it has exactly six actual unordered r-term decompositions. The entire normalization fibre consists of exactly $r+1$ reduced points indexed by the r-element subsets of $\{[Q_1], \dots, [Q_6], [R], [S_1], \dots, [S_k]\}$, as follows:

1. *The six subsets omitting a Q-point give smooth points and the six actual decompositions.*

2. *The subset omitting R gives one singular point with local model $\mathcal{R}_{r-6,a-3}$ and tangent excess $(r-6)(a-3)$.*

3. *Each subset omitting an S_j gives a singular point with local model $\mathcal{B}_{r-7,a-4,b-r+4}$ and tangent excess $(r-7)(a-3)+b-r+4$.*

The latter configurations are dependent, not actual decompositions. All the displayed excesses are positive, including at $r=7$, when $\mathcal{B}_{0,a-4,b-3}$ is $\mathcal{R}_{1,b-3}$ after renaming p, u_h. Every normalization point has dimension $D=r(a+b+c-2)$; its embedding dimension is D at a smooth point and D plus the stated excess at a boundary point. Normalization is unramified over some Zariski open neighbourhood of T. Transposition gives the family with one added column and $r-6$ added rows, with all formulas transported by the swap.

Theorem 7.14 (Two rows and two columns, E2)**.** *Let $a,b\geq 5$, $a+b\geq 11$, and $c\geq 9$. Choose new coordinates u_1,u_2 in A and w_1,w_2 in B, and put*

$$\begin{aligned} R_1 &= u_1\otimes v(4), \qquad R_2 = u_2\otimes v(5),\\ S_1 &= v(4)\otimes w_1, \qquad S_2 = v(5)\otimes w_2,\\ L &= L_0\oplus\langle R_1,R_2,S_1,S_2\rangle,\\ T &= \sum_{i=1}^{5} Q_i\otimes f_i + R_1\otimes f_6 + R_2\otimes f_7 + S_1\otimes f_8 + S_2\otimes f_9. \end{aligned} \tag{7.32}$$

This is $T_{9;5,5}$, with support dimensions $(5,5,9)$, padded to the stated format. Its rank and border rank equal nine, and it has exactly six actual unordered nine-term decompositions. Its entire normalization fibre consists of ten reduced points indexed by the nine-element subsets of the ten displayed matrix directions:

1. *Six smooth points omit a Q-point and correspond to the six actual decompositions.*

2. *The two points omitting R_1 or R_2 are singular, with model $\mathcal{B}_{2,a-4,b-5}$ and excess $2a+b-11$.*

3. *The two points omitting S_1 or S_2 are singular, with model $\mathcal{B}_{2,b-4,a-5}$ and excess $a+2b-11$.*

The four boundary configurations are dependent. Both excesses are at least five. The cases $b=5$ and $a=5$ are included, giving $\beta=0$ in the respective star model. Every normalization point has dimension $D=9(a+b+c-2)$ and embedding dimension D plus its listed excess, with excess zero at the smooth points. Normalization is unramified over a Zariski open neighbourhood of T.

Theorems 7.13 and 7.14 concern exactly the displayed mixed witnesses and their admissible padding. Their conclusions transport under factor permutations with the witnesses. They make no assertion for broader mixed splits with several retained rows and columns, nor any global regularity, Cohen–Macaulayness, rational-singularity or exact-depth assertion about a normalization.

7.9 The configuration graph and the exact dependence ideals

We prove the three theorems together, retaining the exact parameters in each statement. Write $W=A\otimes B$, $S=\mathrm{Seg}(\mathbb{P}A\times\mathbb{P}B)\subset\mathbb{P}W$, and let H be the closure in $\mathrm{Gr}(r,W)$ of the spans of r independent points of S.

Lemma 7.15. *For each displayed witness, $\mathbb{P}(L)\cap S$ is the reduced scheme of its $r+1$ displayed matrix directions. The only relation is the six-core relation. Exactly six r-subsets are independent; the remaining subsets have span dimension $r-1$. The unordered span map is birational onto H, and $\dim H=r(a+b-2)$.*

The reduced closure G of the independent unordered span graph in $\mathrm{Hilb}^r(S)\times H$ is integral, proper and birational over H. Its fibre over L is the reduced scheme of all $r+1$ r-element subsets. Its six independent-configuration points are smooth.

Proof. The core computation of 7.2 gives the reduced six-point intersection in L_0. The avoidance calculation in the row–column construction gives, for every integer $z\geq 4$,

$$L_0\cap(A_0\otimes\mathbb{C}v(z))=0, \qquad L_0\cap(\mathbb{C}v(z)\otimes B_0)=0. \tag{7.33}$$

For the first equality, the determinant of the rows $\mathbf{1}, v(z)^{\mathsf{T}}, v(z)^{\mathsf{T}}/(1,2,3)$ is $-z(z-1)(z-3)/6$, and the remaining third row eliminates the last coefficient. For the second, writing $v(z)b^{\mathsf{T}} \in L_0$ gives $b_j = jd/(j-z)$ and $(j+z)d = je$ for $j = 1,2,3$; the first two equations force $d = 0$ and then $b = 0$. The v-directions are distinct and avoid the core factor directions.

Let E_A, E_B be the added row and column spaces. Every matrix in L has zero $E_A \otimes E_B$ block. For a rank-one matrix $x \otimes y$, this forces $x_{E_A} = 0$ or $y_{E_B} = 0$. A nonzero new row forces the latter equality and its assigned v-direction; distinct directions kill all other new rows, and (7.33) kills the core block. The matrix is that R-direction. Transposition gives the S-directions. If neither new block is present, the matrix lies in L_0. Padding introduces no further possibility: applying the quotient maps shows that a nonzero rank-one matrix in a tensor product of subspaces has both factors in those subspaces.

At a Q-point, a new row or column in the rank-one tangent space has its core factor direction, whereas its possible component in L has a distinct v-direction. All added coefficients vanish, leaving the core scalar tangent line. At an R-point, the zero mixed block kills the varying new-column component; distinct row directions kill the other new rows, and (7.33) kills the core component. Only the scalar tangent remains. The S-case is the transpose, and padding adds separate complementary tangent blocks. The projective tangent spaces therefore vanish. Nakayama's lemma applied to $\mathfrak{m}/\mathfrak{m}^2 = 0$ proves reducedness.

The added matrices occupy distinct row or column blocks, so their only relation is $\sum_i Q_i = 0$. The independent subsets are precisely those omitting a Q-point. The third flattening has rank r and image L. An actual r-term decomposition must therefore choose a basis among these rank-one directions, and that basis determines its third factors uniquely. There are precisely six actual unordered decompositions; flattening rank also proves that ordinary rank and border rank are r. Their addition differentials are injective modulo summand scalings: in an independent third-factor basis, every matrix variation lies both in L and in its rank-one tangent space, so it is scalar; the corresponding third-factor variation cancels exactly that scaling.

For generic factor lists and every subset J of size $j \geq 2$,

$$\operatorname{rank}(a_i : i \in J) + \operatorname{rank}(b_i : i \in J) = \min(a, j) + \min(b, j) \geq j + 2.$$

The last inequality holds because $a, b \geq 3$ and $a + b \geq r + 2$. Sylvester's inequality makes every combination supported on at least two terms have matrix rank at least two. Thus the chosen matrices are independent and their span has exactly its original rank-one directions. The generic unordered configuration is unique, so in characteristic zero the span map is birational. Its domain has dimension $r(a + b - 2)$.

The graph closure is integral and proper over H by projectivity of the Hilbert scheme. The condition $Z \subset \mathbb{P}(K)$ is closed: it is the vanishing of the universal evaluation of the annihilator of the tautological K in the rank-r bundle $H^0(Z, \mathcal{O}_Z(1))$. It holds on the dense independent graph and hence on its reduced closure. Its fibre over L is consequently a closed subscheme of $\operatorname{Hilb}^r(S \cap \mathbb{P}(L))$, the discrete reduced scheme of r-subsets. Repeated or infinitesimal configurations cannot occur, and a pair (Z, L) determines at most one underlying graph point. The six independent subsets give smooth graph points, since there the graph is the graph of the span morphism on the smooth unordered configuration space.

Every dependent subset occurs. If $R_s = u_s \otimes v(s+3)$ is omitted, write $v(s+3) = \sum_{i=1}^{3} \lambda_i r_i^{\mathsf{T}}$, where r_i are the rows of M, and deform

$$Q_i(\epsilon) = (e_i + \epsilon \lambda_i u_s) r_i \quad (1 \leq i \leq 3), \tag{7.34}$$

keeping the other core and retained added matrices fixed. The sum of the six core matrices is ϵR_s. A relation for $\epsilon \neq 0$, projected to the core, makes all six core coefficients equal. The retained added blocks then kill their own coefficients, and the omitted row kills the common coefficient. The matrices are independent. Five deformed core matrices, the retained additions and the fixed R_s form a basis with independent limits, so their spans tend to L.

If $S_s = v(s+3) \otimes w_s$ is omitted, choose η_j with $v(s+3) = \sum_j \eta_j(-m_j)$ and deform instead

$$Q_{3+j}(\epsilon) = (-m_j)(e_j + \epsilon \eta_j w_s)^{\mathsf{T}} \quad (1 \leq j \leq 3). \tag{7.35}$$

Projection to the old core and then to the retained and omitted column blocks gives the same independence and limiting-span argument. These are the arcs of 7.4. Every subset is thus present, proving the asserted reduced fibre. This alone does not exclude additional points after normalization; that requires the local normality proved below. □

Lemma 7.16 (Exact boundary charts)**.** *At the boundary configurations of Theorems 7.12–7.14, the graph germs are smoothly equivalent to the homogeneous origins of the local algebras listed in those theorems.*

Proof. Locally order the distinct points, a free finite étale cover, and choose nonzero factor-vector representatives, a smooth torus torsor. Call the resulting smooth space F. Near such a configuration, the Hilbert-scheme graph is the ordinary graph of the rational span map on the smooth configuration space.

The r representative columns have rank $r-1$. Fix an invertible $(r-1)$-column coordinate minor and let I be the ideal of all $r \times r$ minors. Row elimination identifies I exactly with the residual coordinates of the final column modulo the first $r-1$. Introduce $r-1$ relation coefficients, normalizing one core coefficient to one. The five other core coefficients are units; the coefficients of the retained additions vanish at the configuration. The pivot minor solves the auxiliary coefficients uniquely as regular functions on F. If w_i are their differences from the solved values, the vector-relation ideal on $F \times \mathbb{A}^{r-1}$ is

$$J = (w_1, \dots, w_{r-1}, I). \tag{7.36}$$

Absorb the five unit core coefficients into their first factor vectors. With core matrices A_c, B_c and retained-factor matrices U, V, the relation becomes

$$A_c B_c^{\mathsf{T}} + U \Delta V^{\mathsf{T}} = 0, \tag{7.37}$$

where Δ is diagonal in the retained coefficients. No vanishing coefficient is inverted.

Split the core columns into triples and rows by the old supports:

$$A_c = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_c = [B_1 \ B_2], \quad B_2 = \begin{pmatrix} B_{20} \\ B_{21} \end{pmatrix}, \quad U = \begin{pmatrix} U_0 \\ U_1 \end{pmatrix}, \quad V = \begin{pmatrix} V_0 \\ V_1 \end{pmatrix}.$$

The blocks A_{11}, B_{20} are invertible at the circuit. On the neighbourhood where they remain invertible, use the substitutions

$$\begin{gathered} X = A_{11}^{-1} A_{12}, \quad Y = A_{21} A_{11}^{-1}, \quad P = A_{22} - A_{21} X, \quad C_0 = B_1 + B_2 X^{\mathsf{T}}, \\ U' = U_1 - Y U_0, \qquad V' = V_1 - B_{21} B_{20}^{-1} V_0. \end{gathered} \tag{7.38}$$

These are regular invertible coordinate changes: they solve uniquely for A_{22}, B_1, U_1, V_1, retaining the other coordinates. The top three A rows of (7.37) solve

$$C_0^{\mathsf{T}} = -A_{11}^{-1} U_0 \Delta V^{\mathsf{T}}.$$

Subtracting Y times those rows leaves $P B_2^{\mathsf{T}} + U' \Delta V^{\mathsf{T}}$. Its first three B columns solve

$$P = -U' \Delta V_0^{\mathsf{T}} (B_{20}^{\mathsf{T}})^{-1}.$$

The remaining equations are exactly the entries of

$$U' \Delta (V')^{\mathsf{T}}. \tag{7.39}$$

All operations are invertible operations on ideals. The solved blocks give $d_0 = 3b + 3(a-3) = 3a + 3b - 9$ independent coordinate generators $L_1, \dots, L_{d_0}$.

We identify the residual ideal in each case. For D, there are $\ell = r - 6$ retained rows. Their U' columns are independent new coordinate directions at the witness, so a fixed $\ell \times \ell$ row minor gives a regular left inverse. Multiplication by this inverse identifies the ideal of (7.39) with that of $\Delta (V')^{\mathsf{T}}$; the reverse containment follows by multiplication by U'. With μ_j the coefficients and y_{ij} the entries of V', the ideal is

$$(\mu_j y_{ij} : 1 \le j \le r-6, \ 1 \le i \le b-3). \tag{7.40}$$

For $\ell = 0$ it is zero. The μ, y are independent vanishing coordinates; the retained U' data are smooth parameters.

For E1 with its single row omitted, all $r-6$ retained additions are columns. A regular left inverse of the independent V' columns identifies (7.39) with $U'\Delta$. Its ideal is

$$(q_j y_{ij} : 1 \le j \le r-6, \ 1 \le i \le a-3), \tag{7.41}$$

again in independent vanishing coordinates, with the remaining V' data smooth parameters.

All other cases have one retained row and k retained columns, after transposition if necessary. At the witness $U'_R \neq 0$ and $V'_R = 0$, whereas $U'_{S_j} = 0$ and the V'_{S_j} are independent. Regular invertible operations in A/A_0 send U'_R to e_1; independently, operations in B/B_0 send V'_{S_j} to e_j. The required coordinate and minor are units, and their data remain smooth parameters. These changes act invertibly on the opposite vanishing columns. Write

$$\alpha = a-4, \quad \beta = b-3-k, \qquad U'_{S_j} = (x_j, v_{1j}, \dots, v_{\alpha j})^{\mathsf{T}}, \quad V'_R = (z_1, \dots, z_k, u_1, \dots, u_\beta)^{\mathsf{T}}.$$

If p and q_j are the row and column coefficients, the complete nonzero entry list of (7.39) is

$$pz_j + q_j x_j \ (1 \le j \le k), \qquad pu_h \ (1 \le h \le \beta), \qquad q_j v_{ij} \ (1 \le i \le \alpha, \ 1 \le j \le k). \tag{7.42}$$

All other entries vanish identically. In particular there are no terms $q_j x_l$ for $l \neq j$ and no omitted higher-order remainders. For E1 with a column omitted, $(k, \alpha, \beta) = (r-7, a-4, b-r+4)$, with $\beta \ge 1$; the case $k = 0$ leaves just pu_h. For E2 with a row omitted, $(k, \alpha, \beta) = (2, a-4, b-5)$, where $\alpha \ge 1$ and $\beta \ge 0$; if $\beta = 0$, admissibility gives $\alpha \ge 2$. For a column omitted, transpose to obtain $(2, b-4, a-5)$.

Let $\mathcal{Q}$ denote the relevant quadratic list. We have the exact ideal $J = (L_1, \dots, L_{d_0}, \mathcal{Q})$. We must remove the auxiliary relation coordinates while preserving this embedded model. Every w_i in (7.36) belongs to J. All nonlinear coordinates vanish at the witness, so $d\mathcal{Q} = 0$ there and each dw_i is a linear combination of the dL_j. The $r-1$ differentials dw_i are independent, since they are graph coordinates for the auxiliary coefficients. One coefficient minor is therefore a unit. Express w_i in the displayed ideal generators and replace those $r-1$ linear coordinates by w_i. Inverting that coefficient minor preserves the generated ideal together with the remaining L_j and $\mathcal{Q}$. The Jacobian is invertible while the nonlinear and unused coordinates are retained. Restricting to $w = 0$ returns the original F and gives the exact étale-local ideal

$$I = (l_1, \dots, l_\kappa, \mathcal{Q}), \qquad \kappa = d_0 - (r-1) = 3a + 3b - r - 8. \tag{7.43}$$

The nonlinear variables remain independent. No radical, initial ideal or higher-order truncation has been taken. The number κ counts linear generators and is independent of the smooth-parameter count in the rank-five local-ring calculation.

The span map is given in Plücker coordinates by the maximal minors, so its graph is the blowup $\operatorname{Proj} \bigoplus_{n \ge 0} I^n$. Equivalently, this is the closure of the graph of the projective rational map defined by these generators. Flat base change commutes with ideal powers, so the description is preserved by the étale coordinates and the factor-vector cover.

Use the arc (7.34) or (7.35) for the omitted addition. Choose the pivot from five independent old-core entries and one detecting entry for each retained addition: its own new row with an old B coordinate, or its own new column with an old A coordinate. None uses the omitted row or column. On this arc, the normalized pivot relation is still the six-core relation and all retained coefficients are identically zero. The unchanged old-core entries first determine the six core coefficients; the distinct retained entries then force all other coefficients to vanish. Consequently every residual quadratic is identically zero on the arc, including when an opposite Schur-substituted factor varies to first order. In D, one can also see this from the fact that every B factor stays in B_0, so $V' = 0$ throughout the arc.

The relation residual has nonzero derivative equal to the omitted matrix modulo the retained span. Since $d\mathcal{Q} = 0$, this gives a nonzero linear direction in (7.43). Choose $t = l_1$ nonzero on it and replace the other l_i by combinations whose derivative on the arc is zero. The t-chart is the actual subalgebra

$$\mathcal{O}_F[l_2/t, \dots, l_\kappa/t, \mathcal{Q}/t] \ \subset \ \mathcal{O}_F[t^{-1}]. \tag{7.44}$$

Writing $l_i = th_i$ splits off smooth coordinates h_i, centered at zero at the limit, together with the unused parameters. The remaining algebra is exactly $\mathcal{R}_{r-6,b-3}$ in D, $\mathcal{R}_{r-6,a-3}$ in the omitted-row case of E1, and the stated $\mathcal{B}$ in all star cases. Every model base variable vanishes at the witness. Every nonlinear numerator vanishes along the whole arc, so each ratio by t has limit zero. The point is the homogeneous origin in each case, including opposite omitted and retained types.

The subalgebra (7.44) retains full t-saturation. Equations merely setting t times a fraction equal to its numerator need not present it. For example, when $\ell = 1$ in the monomial case, the additional two-by-two minors of $\left(\begin{smallmatrix} t & y_{11} & \cdots & y_{m1} \\ \mu_1 & \mu_1 y_{11}/t & \cdots & \mu_1 y_{m1}/t \end{smallmatrix}\right)$ are also necessary. The arcs place the selected point in the actual graph closure, without an unsaturated presentation. This proves the asserted smooth equivalences. □

7.10 Normality of the boundary algebras

Proposition 7.17. *For all $s, m \geq 0$, $\mathcal{R}_{s,m}$ is normal. At its homogeneous origin its dimension is $1 + s + sm$, its embedding dimension is $1 + s + 2sm$, and it is smooth precisely when $sm = 0$.*

Proof. Represent exponents by $(K; A_j; B_{ij})$, with K the t exponent. The exponent monoid is exactly the integer vectors satisfying

$$A_j \geq 0, \quad B_{ij} \geq 0, \qquad K + \sum_j \min\{A_j, \sum_i B_{ij}\} \geq 0. \tag{7.45}$$

Indeed, if p_{ij} copies of $\mu_j y_{ij}/t$ occur, then $\sum_i p_{ij} \leq A_j$ and $p_{ij} \leq B_{ij}$, proving necessity. If $K \geq 0$, base variables suffice. If $K < 0$, distribute $-K$ integer units among groups j with capacities $\min\{A_j, \sum_i B_{ij}\}$, and then within each group among the integer capacities B_{ij}. Successive allocation gives integers p_{ij} with total $-K$ and the required bounds. The remaining exponents are supplied by base variables. This proves sufficiency.

The final inequality is equivalent to the finite family

$$K + \sum_{j \in E} A_j + \sum_{j \notin E, i} B_{ij} \geq 0 \qquad (E \subset \{1, \ldots, s\}).$$

The generator lattice is the full integer lattice because all base variables occur. Thus the monoid is its rational cone intersected with that lattice. More explicitly, intersect the Laurent polynomial ring in all base variables with the monomial-weight valuation rings for these inequalities and the nonnegativity inequalities. A weight valuation of a Laurent polynomial is the minimum of its monomial weights; it is multiplicative because lowest homogeneous parts multiply nontrivially in the Laurent polynomial domain. The intersection consists exactly of Laurent polynomials whose exponents satisfy (7.45), hence is $\mathcal{R}_{s,m}$. Each ring in the intersection is integrally closed, proving normality, including empty index ranges.

The total-degree functional $K + \sum_j A_j + \sum_{i,j} B_{ij}$ gives every displayed generator degree one. They are distinct Laurent monomials, so they have no linear relation and form a basis of the cotangent space at the origin. There are $1 + s + sm$ independent base variables and sm fractions. The fraction field has transcendence degree $1 + s + sm$, while the cotangent dimension is $1 + s + 2sm$. For $sm = 0$ the ring is polynomial; otherwise the positive tangent excess makes the origin singular. □

Lemma 7.18 (Regular-parameter deformation)**.** *Let D be a local domain obtained by localizing a finitely generated $\mathbb{C}$-domain, and let $0 \neq f$ lie in its maximal ideal. If D/fD is a normal domain, then D is normal.*

Proof. The normalization N of D is finite, since finite-type $\mathbb{C}$-domains have finite normalization and normalization commutes with localization. The prime ideal (f) has height one by the principal ideal theorem: D is a domain and f is nonzero and not a unit. Hence $D_{(f)}$ is a one-dimensional local domain with principal maximal ideal, so it is a discrete valuation ring. Every element of N is integral over $D_{(f)}$, whence $N \subset D_{(f)}$. Localizing N at $D \setminus (f)$ gives $D_{(f)}$ itself. There is exactly one prime Q of N above (f), with $N_Q = D_{(f)}$, and f has valuation one at Q.

Every height-one prime of N containing f contracts to (f). Indeed, a finite birational extension of affine $\mathbb{C}$-domains preserves prime heights: total dimensions agree and the corresponding residue fields have the same transcendence degree, since their extension is finite. The dimension formula gives equality of heights, which persists under the localizations defining D, N. A height-one contraction containing (f) must equal (f). Thus Q is the only height-one prime of N containing f.

Reduction in $N_Q = D_{(f)}$ embeds N/Q in $\mathrm{Frac}(D/fD)$. It is integral over the normal domain D/fD, so $N/Q = D/fD$. Given $n \in N$, choose $d \in D$ with the same reduction modulo Q. The fraction $(n - d)/f$ has nonnegative valuation at Q, because the numerator has valuation at least one. At every other height-one prime, f is a unit, so the valuation is again nonnegative. A Noetherian normal domain is the intersection of its height-one valuation rings in its fraction field; hence $(n - d)/f \in N$. Therefore $N = D + fN$. The finite D-module N/D satisfies $N/D = f(N/D)$, and Nakayama's lemma gives $N/D = 0$. □

Corollary 7.19. *Let E be a finitely generated nonnegatively graded $\mathbb{C}$-domain, and let ξ be a nonzero homogeneous element of positive degree. If $E/(\xi)$ is a normal domain, then E is normal.*

Proof. Lemma 7.18 proves normality at every prime containing ξ. The nonnormal locus is closed and invariant under the multiplicative action of the grading. Choose finitely many homogeneous algebra generators, all of

nonnegative degree. Every closed point has an affine limit under scalar multiplication as the scalar tends to zero: positive-degree coordinates tend to zero and degree-zero coordinates remain fixed. The limit lies on $\xi = 0$. A nonempty nonnormal locus would contain a closed point, its nonzero-scalar orbit and its limit, contradicting normality on that fibre. The nonnegative grading is used here; normality along one fibre of an arbitrary nonproper family would not suffice. □

Proposition 7.20 (Star algebra)**.** *For every $k, \alpha, \beta \geq 0$, the exact subalgebra $\mathcal{B}_{k,\alpha,\beta}$ of (7.29) is normal. Its dimension and the embedding dimension at its homogeneous origin are*

$$d = 2 + 3k + \beta + \alpha k, \qquad N = 2 + 4k + 2\beta + 2\alpha k.$$

The tangent excess is $k(\alpha + 1) + \beta$. The origin is smooth precisely when $k = \beta = 0$.

Proof. Write $R = \mathcal{B}_{k,\alpha,\beta}$ and abbreviate its fractions by

$$G_j = \frac{pz_j + q_j x_j}{t}, \qquad P_h = \frac{pu_h}{t}, \qquad Q_{ij} = \frac{q_j v_{ij}}{t}.$$

The additional elements

$$H_{ij} = \frac{pz_j v_{ij}}{t} = v_{ij} G_j - x_j Q_{ij} \tag{7.46}$$

belong to R. Let $B = \mathbb{C}[t, p, q, x, z, u, v, P, Q, H]$, a monomial algebra; then $R = B[G_1, \dots, G_k]$. Give q_j degree one and all other free variables degree zero. Thus Q_{ij} has degree one, P_h, H_{ij} degree zero, and the leading part of G_j is $E_j = q_j x_j / t$. We prove

$$\operatorname{gr} R = A := B[E_1, \dots, E_k] \tag{7.47}$$

for this total-q filtration.

For $S_0 \subset \{1, \dots, k\}$, put $B_{S_0} = B[E_l : l \in S_0]$ and fix $j \notin S_0$. The only fractional generators consuming q_j in this monomial algebra are Q_{ij}, and x_j occurs only as a free variable. These facts remain true at every intermediate stage. For each $n \geq 0$,

$$\{b \in B_{S_0} : bE_j^n \in B_{S_0}\} = (t, v_{1j}, \dots, v_{\alpha j})^n. \tag{7.48}$$

To prove this, first let b be a Laurent monomial and choose a generator representation of bE_j^n. Let s be its number of Q_{ij} factors with this fixed j, counted with multiplicity. If b has q_j, x_j exponents a_q, a_x, the representation has exactly $a_q + n - s$ free q_j factors and $a_x + n$ free x_j factors; both a_q, a_x are nonnegative. If $s \geq n$, remove n such Q factors and n free x_j factors. Their product is $q_j^n x_j^n / t^n$ times a degree-n monomial V in the v_{ij}. The remaining expression is $b/V \in B_{S_0}$, proving membership in the right side. If $s < n$, remove all s such Q factors, $n - s$ free q_j factors and n free x_j factors. There are enough free q_j factors since $a_q + n - s \geq n - s$. The remaining expression is $b/(t^{n-s}V) \in B_{S_0}$, with V a degree-s v-monomial, again proving membership. Conversely,

$$tE_j = q_j x_j, \qquad v_{ij} E_j = x_j Q_{ij}$$

belong to B_{S_0}, so every product of n of these ideal generators belongs to the left side. Multiplication by E_j^n preserves distinct Laurent monomials, making membership termwise. This proves the full ideal equality, including $\alpha = 0$, when the right side is (t^n).

The complete relative kernel is

$$\ker\big(B_{S_0}[Y_j] \longrightarrow B_{S_0}[E_j]\big) = (tY_j - q_j x_j,\ v_{ij} Y_j - x_j Q_{ij} : 1 \leq i \leq \alpha). \tag{7.49}$$

Group polynomial terms by equal Laurent images. Every kernel element is a linear combination of binomials $bY_j^n - cY_j^m$. Cancelling a common Y_j power reduces to $bY_j^e - c$, where $c = bE_j^e \in B_{S_0}$. By (7.48), some product $g_1 \cdots g_e$ with each $g_l \in \{t, v_{1j}, \dots, v_{\alpha j}\}$ divides the monomial b in B_{S_0}. Write $b = b' g_1 \cdots g_e$ and $h_l = g_l E_j$. Each h_l is $q_j x_j$ or $x_j Q_{ij}$. The product telescoping identity expresses

$$b'\big((g_1 \cdots g_e) Y_j^e - h_1 \cdots h_e\big)$$

as a combination of $g_l Y_j - h_l$. Multiplying by the cancelled power proves (7.49) before localization, including higher powers and torsion relations.

Adjoin the E_j in order. At each stage the displayed relative kernel generators have coefficients in the original B. Their preimage in the preceding polynomial presentation adds precisely these generators to the known kernel. Induction gives

$$\ker\big(B[Y_1,\dots,Y_k]\longrightarrow A\big)=(tY_j-q_jx_j,\ v_{ij}Y_j-x_jQ_{ij}:\text{all } i,j). \tag{7.50}$$

Thus cross-relations among several Y_j are included; relations internal to B are already identities in the coefficient algebra.

Give every Y_j weight one in addition to the q-grading on B. The generators in (7.50) have weight one, while substitution of G_j gives

$$tG_j-q_jx_j=pz_j,\qquad v_{ij}G_j-x_jQ_{ij}=H_{ij}, \tag{7.51}$$

whose right sides have weight zero. Represent $f\in R$ by $F\in B[Y_1,\dots,Y_k]$ and let d be its largest weight. If its weight-d component survives substitution of E, it is the leading part of f and lies in A. If it cancels, it belongs to (7.50). Express it using homogeneous coefficients of weight $d-1$ times the weight-one kernel generators, and use (7.51) after substitution of G. This represents the same f with strictly smaller largest weight. Nonnegative integer weights ensure termination. Every leading part therefore lies in A; conversely its generators are leading parts of the corresponding generators of B or G_j. This proves (7.47). The same reduction shows that any f of degree at most d has a representation of weighted degree at most d in the enlarged family including the weight-zero H_{ij}.

We now prove saturation of the monomial algebra A. Write its candidate exponent vector as t exponent K and nonnegative resource exponents $a_p,a_{q_j},a_{x_j},a_{z_j},a_{u_h},a_{v_{ij}}$. The negative-$t$ generators P_h,E_j,Q_{ij},H_{ij} can have nonnegative multiplicities $P'_h,E'_j,Q'_{ij},H'_{ij}$ precisely subject to the resource bounds

$$\begin{gathered}\sum_h P'_h+\sum_{i,j}H'_{ij}\le a_p,\qquad E'_j+\sum_i Q'_{ij}\le a_{q_j},\qquad E'_j\le a_{x_j},\\ \sum_i H'_{ij}\le a_{z_j},\qquad P'_h\le a_{u_h},\qquad Q'_{ij}+H'_{ij}\le a_{v_{ij}}.\end{gathered} \tag{7.52}$$

If N' is their total multiplicity, the required t exponent is possible exactly when $N'\ge -K$, using $K+N'$ free t factors. Unused resources are supplied by free variables.

The bounds are a single-commodity flow network. Its source has edges to p and q_j of capacities a_p,a_{q_j}. There are unrestricted edges $p\to u_h$, followed by u_h-to-sink edges of capacity a_{u_h}. An edge $p\to \mathrm{gate}_j$ has capacity a_{z_j}, followed by unrestricted edges to v_{ij}. There are unrestricted edges $q_j\to x_j$ and $q_j\to v_{ij}$; their edges to the sink have capacities a_{x_j} and $a_{v_{ij}}$, respectively. The four path types are P,H,E,Q. Every unrestricted edge may be given finite capacity $a_p+\sum_j a_{q_j}$. For integer resource data all capacities are integral, so integral max-flow supplies integer multiplicities whenever real multiplicities reach the integer demand $-K>0$. Deleting unit paths permits any smaller required nonnegative integer flow. If $K\ge 0$, free variables already give a representation.

The exponent cone is exactly this real-feasible region: a real generator expression gives a packing, and a packing with unused resources gives a real generator expression. Every integer point of the cone consequently has an integer generator expression. The lattice is the full integer lattice because all free variables are present. Thus the exponent monoid is its cone intersected with the lattice. Intersecting the Laurent polynomial ring with the monomial valuation rings for the rational cone facets, as in Proposition 7.17, proves that A is normal. This argument includes all empty index sets.

Let F_dR denote elements of total q degree at most d, and form the weighted Rees algebra

$$\mathcal{E}=\bigoplus_{d\ge 0}F_dR\,\xi^d.$$

The terminating reduction proves finite generation: generators are ξ, the weight-zero elements t,p,x,z,u,v,P,H, and $q_j\xi,Q_{ij}\xi,G_j\xi$. Including H_{ij} explicitly is essential, since its cancellation expression (7.46) has larger weight before cancellation. The grading is nonnegative, and

$$\mathcal{E}/(\xi)=\operatorname{gr}R=A,\qquad \mathcal{E}[\xi^{-1}]=R[\xi,\xi^{-1}].$$

Corollary 7.19 makes $\mathcal{E}$ normal. Localizing gives a normal $R[\xi,\xi^{-1}]$. An element of $\mathrm{Frac}(R)$ integral over R belongs to that Laurent polynomial ring and, being independent of ξ, belongs to R. Thus R is normal.

Finally, inverting t eliminates all fraction generators, so the fraction field is that of its $2+3k+\beta+\alpha k$ independent base variables. For the homogeneous origin, now give *all* base variables degree one. The original G, P, Q also have degree one, while H has degree two. The degree-zero part is $\mathbb{C}$. There is no linear relation among the original generators: free variables and P, Q are distinct Laurent monomials, and the monomial pz_j/t occurs in the unique generator G_j. Its coefficient first forces each G_j coefficient to vanish; the remaining monomial coefficients then vanish. Hence the original degree-one generators form a basis of the cotangent space. Their number is $2+3k+\beta+\alpha k+(k+\beta+\alpha k)=N$. The excess is $k+\beta+\alpha k$, positive unless $k=\beta=0$. In that case $R=\mathbb{C}[t,p]$, independently of α, and is smooth. □

7.11 Identification of the entire fibres and their local dimensions

Proof of Theorems 7.12, 7.13 and 7.14. Lemma 7.15 gives precisely the stated graph configurations, six of them smooth. Lemma 7.16 gives the exact local models at every dependent configuration, and Propositions 7.17 and 7.20 prove them normal. Normality descends through the factor-vector torus torsor and the free ordered-configuration étale quotient. Thus G is normal at every point of its finite fibre over L.

The non-quasi-finite locus is closed and misses that fibre; remove its proper image in H. The nonnormal locus is likewise closed for a variety over $\mathbb{C}$, and its proper image also misses L. After these shrinkings, $G \to H$ is finite with normal integral source and is birational. It is therefore the entire normalization: on affine rings, a finite normal extension with the same fraction field equals the integral closure. In particular there is precisely one normalization point at each of the $r+1$ configurations, or ten configurations when $r=9$ in E2.

We also check unramifiedness scheme-theoretically. The projective point incidence $Z_H=\{(K,p) : p \in \mathbb{P}(K)\cap S\}$ has finite reduced fibre at L. Properness makes it finite near L. Its fibre cotangent spaces vanish; base change for relative differentials and Nakayama's lemma give unramifiedness at the fibre. Removing the closed image of the remaining non-unramified locus gives a finite unramified incidence nearby. In its r-fold fibre product the distinct-point locus is open and closed, since diagonals of a separated unramified morphism are open and closed. This locus is therefore finite and unramified over H.

The graph with its points ordered is a closed subscheme of that incidence near the fibre. The proper image of the locus of nonreduced Hilbert-scheme configurations may first be removed, since it misses the fibre. Closed immersions are unramified, and descent through the free étale permutation quotient proves unramifiedness of $G \to H$. Over $\mathbb{C}$, its finite unramified fibres are reduced. When the models in D are smooth, all points over L are smooth; removing the proper image of the nonsmooth locus gives a neighbourhood whose entire normalization source is smooth. We have used only the incidence and bundle constructions underlying 7.1; its all-subsets-independent hypothesis is not imposed or used at these dependent configurations.

On the rank-r locus of the third flattening, $X_r(a,b,c)$ is the bundle of surjections $(\mathbb{C}^c)^* \to K$ over $K \in H$. For an independent rank-one span, every such surjection is an r-term tensor sum, and conversely a rank-r-flattening r-term sum gives such a span. Taking reduced closures preserves this equality: a local trivialization is a product, and the bundle projection is smooth and open. Pulling the normalization of H back along this bundle gives a finite birational source that is an open vector bundle over a normal variety, hence normal. It is the full normalization of this tensor locus. Base change preserves unramifiedness and the indicated fibres; the bundle adds only rc smooth parameters. Thus every point has dimension

$$\dim H + rc = r(a+b+c-2).$$

Smooth and étale covers preserve tangent excess. For D, Proposition 7.17 gives $(r-6)(b-3)$, with zero exactly when $r=6$ or $b=3$. For the mixed cases, the excesses are

configuration	tangent excess
E1, omitted row	$(r-6)(a-3)$
E1, omitted column	$(r-7)(a-3)+b-r+4$
E2, omitted row	$2a+b-11$
E2, omitted column	$a+2b-11.$

The first is positive since $r \geq 7, a \geq 4$; the second is positive since $b-r+4 \geq 1$. The last two are at least five because $a, b \geq 5$ and $a+b \geq 11$. These are normal singular points, including the $k=0$ and $\beta=0$ boundary cases allowed by Proposition 7.20. Adding the dimension gives exactly the claimed embedding dimensions, and

the six actual-decomposition points remain smooth. Factor permutations transport the variety, normalization and witness, proving the remaining assertions. □

7.12 Local inverse branches and two algebraic consequences

For completeness, the immersive actual decompositions also give a precise local stability statement. If $f : Y \to V$ is holomorphic, Y is a smooth complex manifold of dimension d, V is a finite-dimensional Hermitian vector space, and df_y is injective, choose a linear projection $\pi : V \to \mathbb{C}^d$ for which $d(\pi f)_y$ is invertible. The holomorphic inverse function theorem gives a neighbourhood U on which πf is biholomorphic. Then $(\pi f)^{-1}\pi$ restricts to an inverse of f on $f(U)$. On a smaller relatively compact neighbourhood its derivative is bounded, by continuity. Equivalently, for any fixed smooth local Hermitian metric on Y, the least singular value of df is bounded below by a positive constant there. Each actual decomposition branch above has this property. It implies no probability-tail or moment assertion and no global single-valued inverse.

A retract of a normal domain is normal. Indeed, let $A \hookrightarrow B \xrightarrow{\rho} A$ have identity composite and let B be integrally closed in its fraction field. An element $z \in \mathrm{Frac}(A)$ integral over A is integral over B, so lies in B. Write $z = a/s$ with $a, s \in A$, $s \neq 0$. Applying ρ to $sz = a$ gives $s\rho(z) = a$, and hence $z = \rho(z) \in A$. Coordinate inclusions and projections preserve reduced tensor secants and compose to the identity. Their coordinate rings form such a retract, so nonnormality propagates to larger factor dimensions. This argument does not transfer Cohen–Macaulayness or S_2 through a general ring retraction.

Finally, if $A \subseteq B$ is a finite normalization, then

$$\mathrm{Ann}_A(B/A) = \{a \in A : aB \subseteq A\} = (A : B).$$

The support of a finite module is the vanishing locus of its annihilator. At a prime $\mathfrak{p}$, the quotient vanishes exactly when $B_\mathfrak{p} = A_\mathfrak{p}$; coherence then gives an open isomorphism neighbourhood for the finite normalization. Its nonisomorphism locus is therefore precisely the support of B/A.

8 Exact arithmetic and the finite formal certificate

The fixed finite numerical input consists of the matrix (4.1), its six rank-one row and column matrices, and the six rational values in (4.11) and (4.12). All entries in the tensor and both decompositions are integers. The argument in Section 4 proves the tangent ranks from these determinants without floating-point rank decisions.

8.1 Independent exact arithmetic checks

Two separate exact implementations verify the explicit data. One reads the two lists of integer factors; the other constructs them directly from (4.1). They verify the equality of the resulting tensors, nonzero and distinct summands, distinct decompositions, and the Jacobian ranks 45 and 80. The first additionally checks pairwise distinct projective directions, all three pair-product ranks, and the rank-five third flattening explicitly. The coordinate Jacobian in the smaller format has 45 rows and $5(3+3+5) = 55$ columns; after padding it has 216 rows and $5(6+6+6) = 90$ columns.

Exact elimination is performed modulo the primes 1,000,003 and 101, respectively. A nonzero minor modulo a prime is a nonzero integer minor, so these computations give rigorous lower bounds for the ranks over $\mathbb{Q}$ and $\mathbb{C}$. The prime used by the first implementation is itself verified by integer trial division. For the matching upper bounds, each nonzero pure summand has two independent scaling directions in the factor-coordinate kernel, for example

$$(\dot{a}_i, \dot{b}_i, \dot{c}_i) = (a_i, -b_i, 0), \qquad (a_i, 0, -c_i).$$

Across five summands these give ten independent kernel vectors. The bounds are therefore $55 - 10 = 45$ and $90 - 10 = 80$, matching the modular lower bounds.

The verification includes altered-data tests that break tensor equality, pair-product independence, or distinctness of the decompositions. These are required to be rejected. The positive checks and the altered-data checks have identical prescribed behavior with and without Python optimization, and with two choices of hash seed. All decisive arithmetic uses integers, rational numbers, or finite prime fields.

8.2 The Lean statement and its limits

The accompanying Lean development proves a single finite certificate whose assertions are the following:

1. The two integer factor lists have the same sum, both in format $3 \times 3 \times 5$ and after padding to $6 \times 6 \times 6$.
2. In each list the summands are nonzero and pairwise nonproportional, and the two lists give different sets of directions; these assertions also hold after padding.
3. The two third-mode flattening matrices, before and after padding, have rank five over $\mathbb{C}$.
4. The six determinants are exactly the rational numbers in (4.11) and (4.12).
5. Both coordinate Jacobians have rank 45 over $\mathbb{C}$ in the smaller format and rank 80 over $\mathbb{C}$ after padding.

The formal rank lower bounds use integer right-inverse identities with scale factor 36. The ten explicit scaling directions supply the upper bound for the padded Jacobian. The formalization uses Lean and Mathlib, with the certificate fixed by the displayed integer data.

These are statements about concrete arrays, determinants and ranks. The formal theorem does not state a tensor border-rank bound, identify the geometric tangent space, prove the dimension of a secant variety, or establish any fact about normalization, conductors, S_2, Cohen–Macaulayness, or vertex depth. The passage from the finite certificate to Theorem 1.1, including the uniform extension to all ranks and formats in (1.2), is the written proof in the preceding sections. In particular, neither the tangent ranks alone nor nonnormality alone imply the Cohen–Macaulay conclusion.

8.3 The uniform construction

The extension to arbitrary r uses the identities in Lemma 4.4 and the block comparisons in Lemma 4.5. Identity (4.15) can also be verified coefficient by coefficient in $\mathbb{Q}[t]$; its nonvanishing for every integer $t \geq 4$ is an immediate consequence of the displayed factorization. Thus no search for witnesses or recognition of numerical data is needed as r grows.

For additional finite corroboration, exact arithmetic checks the 15 pairs (r, p) with $5 \leq r \leq 9$ and $3 \leq p \leq r-2$, putting $q = r+1-p$. Each seed in format $p \times q \times r$ and each padded example in format $p \times (q+1) \times r$ is checked, as is the extension from $4 \times 5 \times 8$ to $5 \times 5 \times 8$. These are 31 pairs of decompositions. Equality and distinctness are tested over the integers, while pair-product and Jacobian ranks are checked modulo 101, with the displayed scaling kernels giving matching characteristic-zero upper bounds. Altering equality, distinctness, independence, or an avoidance vector is required to cause rejection, with the same behavior under Python optimization. These checks concern exactly the listed instances. The all-r statement, the smoothness of the incidence charts, the conductor argument, and the local depth conclusions rest on their written proofs.

Statements and Declarations

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Data availability. The exact verification scripts, certificates and Lean 4 files supporting the finite computations are available from the corresponding author on reasonable request.

Use of generative AI and AI-assisted technologies. During the preparation of this work, the authors used AI tools (Claude, Anthropic; Codex, OpenAI; Qwen APP, Alibaba) to generate candidate constructions and code, perform computations, draft mathematical arguments, and improve the readability of the text. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the

content of the publication. The fixed finite witness calculations are certified by exact arithmetic and by a Lean 4 formalization (Section 8); the infinite family and the geometric and depth conclusions follow from the written proofs.

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