Asymptotic Tensor-Rank Sublevels Need Not Be Irreducible

Sketch of proof

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The question and our answer. For a tensor T, write $\widetilde{\mathrm{R}}(T) = \lim_m \mathrm{R}(T^{\boxtimes m})^{1/m}$ for its asymptotic rank, where in the power $T^{\boxtimes m}$ the corresponding modes are grouped. Christandl, Hoeberechts, Nieuwboer, Vrana and Zuiddam proved that every sublevel $\{T : \widetilde{\mathrm{R}}(T) \le a\}$ is Zariski closed [1], and asked whether it is always *irreducible*, for every order $k \ge 3$ (catalogue problem TR-23 [4]). We show that the answer is no:

$$\boxed{Y = \{T \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2 : \ \widetilde{\mathrm{R}}(T) \le \sqrt[3]{62}\} \ \text{ is reducible.}}$$

The counterexample has order four; the order-three question stays open. The plan: find two tensors in Y that "look different" (Steps 1–4), then show that Y splits into two proper closed pieces, one containing each tensor (Steps 5–6).

1. The two witnesses

Let e_0, e_1, e_2 be the standard basis of $\mathbb{C}^3$ and d_0, d_1 that of $\mathbb{C}^2$. Put

$$P = \sum_{(i,j,k) \text{ a permutation of } (0,1,2)} e_i \otimes e_j \otimes e_k, \qquad \boxed{x = P \otimes d_0, \qquad y = e_0^{\otimes 3} \otimes d_0 + e_1^{\otimes 3} \otimes d_1.}$$

The tensor y has rank and asymptotic rank exactly 2, so $y \in Y$. Appending a fixed vector d_0 does not change asymptotic rank, so $x \in Y$ as soon as $\widetilde{\mathrm{R}}(P)^3 < 62$. Proving this strict bound is the main analytic work. Note that x has fourth-mode rank one, while y has fourth-mode rank two.

2. Two classical tools

We use the *asymptotic spectrum* of all complex three-mode tensors: spectral points φ are additive under $\oplus$, multiplicative under $\boxtimes$, normalised by $\varphi(D_r) = r$, and monotone under restriction. Strassen's theorem [3, 2] gives

$$\boxed{\widetilde{\mathrm{R}}(T) = \max_{\varphi} \varphi(T), \qquad \varphi(T) \ge \widetilde{Q}(T).} \tag{1}$$

The second tool is the tight-support theorem [2, Thm. 4.4, Cor. 4.5]: if the support of a three-mode tensor admits injective integer weights summing to zero on every support triple, then for every probability distribution p on the support,

$$\widetilde{Q}(T) \ \ge \ \min_j \exp H(p_j). \tag{2}$$

Monotonicity also holds under explicit (Laurent) degenerations: by interpolation, the constant coefficient of a polynomial family of restrictions is a restriction of a direct sum of $dm+1$ copies, and taking m-th roots removes this factor.

3. The small Coppersmith–Winograd tensor

For even q, the paired tensor C_q is equivalent to the small Coppersmith–Winograd tensor $\mathrm{cw}_q = \sum_{j=1}^{q}(e_0\otimes e_j\otimes e_j + e_j\otimes e_0\otimes e_j + e_j\otimes e_j\otimes e_0)$. The classical $(q+2)$-term degeneration

$$t^{-2}\sum_{j=1}^{q}(e_0+te_j)^{\otimes 3} - t^{-3}\Big(e_0+t^2\sum_j e_j\Big)^{\otimes 3} + (t^{-3}-qt^{-2})e_0^{\otimes 3} \xrightarrow{t\to 0} \mathrm{cw}_q$$

shows $\boxed{\varphi(C_8)\le 10}$ for every spectral point.

4. A finite Fourier exchange and the bound $\widetilde{\mathrm{R}}(P)^3 < 62$

Let $G=(\mathbb{F}_2^2)^3$ (so $|G|=64$), $S=\{1,2,3\}^3$ and $b=(1,1,1)$. The group tensor $K_G=\sum_{u+v+w=0} f_u\otimes f_v\otimes f_w$ has rank at most 64 by Walsh–Fourier orthogonality. Apply in each mode the Laurent map

$$f_g \;\mapsto\; \mathbf{1}_{g\in S}\,x_g + t^{-2}\mathbf{1}_{g=b}\,z_0 + t\,\mathbf{1}_{g\in E}\,z_g, \qquad E = G\setminus(S\cup(S+b)).$$

The constant terms produce $P^{\boxtimes 3}$ (three nonzero elements of $\mathbb{F}_2^2$ sum to zero exactly when they are the three distinct ones) together with a copy of C_{18}; the only negative-degree terms are cancelled by an adjoined copy of C_8. This gives the degeneration

$$\boxed{P^{\boxtimes 3}\oplus C_{18} \;\trianglelefteq\; D_{64}\oplus C_8.} \tag{3}$$

Giving the basis of C_{18} the weights $0,\pm1,\dots,\pm9$ makes its support tight; the uniform distribution on its 54 support triples has marginals $1/3$ at zero and $1/27$ elsewhere, so $\widetilde{Q}(C_{18}) \ge 3\sqrt[3]{81} > 12$. Applying any spectral point to (3):

$$\varphi(P)^3+\varphi(C_{18}) \le 64+\varphi(C_8) \le 74 \quad\Longrightarrow\quad \boxed{\widetilde{\mathrm{R}}(P)^3 \le 74 - 3\sqrt[3]{81} < 62.}$$

Only the integer comparisons $81>64$ and $62<64$ are needed; no numerics.

5. An open set forced into fourth-mode rank one

Write a tensor in V as $T \otimes d_0 + S \otimes d_1$ with $T, S \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Let Z be the locus where the three balanced 9×6 flattenings have rank at most 3, and $X = \{T \otimes v\}$ the closed locus of fourth-mode rank at most one. When the mode flattenings of T have rank 3, the condition "rank ≤ 3" says that every column of S lies in the column space of T. Writing this with 4×4 minors gives 1134 linear equations $L(T)S = 0$, always satisfied by $S = T$. At $T = P$ these equations force S to be symmetric with zero entries on repeated indices, so

$$\boxed{\ker L(P) = \mathbb{C}P, \qquad \operatorname{rank} L(P) = 26.} \tag{4}$$

Let U be the open set where the mode flattenings of T have rank 3 and $\operatorname{rank} L(T) \geq 26$. Then $x \in U$, and on U the kernel is $\mathbb{C}T$, so $S = \lambda T$ and the tensor equals $T \otimes (d_0 + \lambda d_1)$. That is,

$$\boxed{Z \cap U \subseteq X.} \tag{5}$$

This is a statement in the whole 54-dimensional space, not on a slice.

6. The closed cover

Every flattening rank is at most the asymptotic rank, and $\sqrt[3]{62} < 4$, so $Y \subseteq Z$. Put

$$\boxed{Y_1 = Y \cap X, \qquad Y_2 = Y \cap (V \setminus U).}$$

Both are relatively closed in Y, and by (5) they cover Y. They are proper: $y \notin Y_1$ (its fourth-mode rank is two) and $x \notin Y_2$ (since $x \in U$). Hence Y is reducible.

More components. The same mechanism scales. For $n \geq 3$ the concise cores $T_{n,s} = \sum_{i=1}^{n} e_i^{\otimes 3} + s(\sum_i e_i)^{\otimes 3}$ have rank and border rank $n+1$ but asymptotic rank at most $B_n = \big((n+1)^n + n^{2n/3} - ((n+1)^n - n^n)^{2/3}\big)^{1/n} < n+1$. For order $k \geq 4$ and $B_n \leq a < n+1$, the sublevel in format $(d_1, \ldots, d_k)$ has at least $\binom{t_n}{3} + 1$ irreducible components, $t_n = \#\{i : d_i \geq n\}$. By contrast, grouping two modes does not carry reducibility over: the full sublevel in format $(9, 3, 2)$ is an irreducible determinantal variety for $3 \leq a < 4$.

References

[1] M. Christandl, K. Hoeberechts, H. Nieuwboer, P. Vrana, J. Zuiddam. *Asymptotic tensor rank is characterized by polynomials.* arXiv:2411.15789v2; STOC 2025. Theorem 1.2 and Section 5.

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[4] Open Problems in Numerical Linear Algebra, problem TR-23. https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/tensor-computations/TR-23.