Two Vertices Certify a Whole Interval of Nonsingular Sign-Regular Matrices

Sketch of proof

Wenqi Zhu[1], Kai Zou[2] and Ping Nie[3]

[1]Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK. wenqi.zhu@maths.ox.ac.uk
[2]Mathematical & Engineering Sciences, King's College London, UK. kz@netmind.ai
[3]David R. Cheriton School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada.
ping.nie@uwaterloo.ca

The conjecture and our answer. Let $\mathcal{K} = [L, U] = \{X \in \mathbb{R}^{n\times n} : L \le X \le U\}$ entrywise. A matrix X is *sign regular* with signature $\varepsilon \in \{\pm 1\}^n$ if

$$\varepsilon_k \det X[I, J] \ge 0 \qquad (|I| = |J| = k,\ 1 \le k \le n),$$

and *nonsingular sign regular* if moreover $\det X \neq 0$; zero proper minors are allowed. The two *checker vertices* are

$$A^-_{ij} = \begin{cases} L_{ij}, & i+j \text{ even} \\ U_{ij}, & i+j \text{ odd} \end{cases} \qquad A^+_{ij} = \begin{cases} U_{ij}, & i+j \text{ even} \\ L_{ij}, & i+j \text{ odd.} \end{cases}$$

Garloff showed in 1982 that these two vertices certify *strict* sign regularity of the whole box [1]; whether they also certify nonsingular sign regularity in general, with zero minors and fixed entries of both parities allowed, has been open since then (Garloff–Adm–Titi, Conjecture 3.1 [2]; catalogue problem IV-01 [3]). We prove it:

> every $X \in \mathcal{K}$ is nonsingular sign regular with signature $\varepsilon \iff A^-$ and A^+ are.

Only "⇐" needs proof. We induct on the minor order k. A *bad* k-minor means $\varepsilon_k \det X[I, J] < 0$; a zero minor is never a contradiction by itself, so every contradiction below is with a *nonzero* determinant or with nonsingularity.

The mechanism on a 5×5 example. Take $M \in \operatorname{ri}\mathcal{K}$ with rows $1, 2, 3$ written as $[\,a\ u\ s\ v\ c\,]$ and suppose orders $1, 2$ are fine but $B_{\text{bad}} = M[\{1,2,3\}, \{1,3,5\}] = [\,a\ s\ c\,]$ is a bad three-minor of minimal span. A three-term identity (Step 3) forces the two-minors on columns $\{2,3\}$ and $\{3,4\}$ to vanish, so the middle rectangle $[\,u\ s\ v\,]$ has rank one. Interpolation (Step 4) makes $\operatorname{rank} X[\{1,2,3\}, \{2,3,4\}] \le 1$ for *every* $X \in \mathcal{K}$. Step 5 then shows that either $[\,a\ s\,]$ or $[\,s\ c\,]$ has rank ≤ 1 at M, so $\det B_{\text{bad}} = 0$: contradiction. The general proof is this argument, with a Schur complement reducing a bad k-minor to a bad three-minor.

1. Nonsingularity and the initial minor orders

Role. From the two vertices alone, get nonsingularity on the whole box and the signs of orders $1, 2, n-1, n$; this starts the induction and supplies the nonsingularity used later.

Put $D = \operatorname{diag}((-1)^1, \dots, (-1)^n)$ and $\sigma = \varepsilon_{n-1}\varepsilon_n$. The cofactor formula gives $\sigma D(A^{\pm})^{-1}D \geq 0$, and the matrices σDXD, $X \in \mathcal{K}$, form an ordinary entrywise interval with inverse-positive endpoints. The interval fact needed is short: if $P \leq C \leq Q$ with $P^{-1}, Q^{-1} \geq 0$, set $v = P^{-1}\mathbf{1} > 0$, $B = Q^{-1}(Q-P) \geq 0$; then $Bv = v - Q^{-1}\mathbf{1} < v$ gives $\rho(B) < 1$, and $0 \leq E = Q^{-1}(Q-C) \leq B$ shows $C = Q(I-E)$ is invertible with $C^{-1} \geq 0$. Hence

$$\boxed{\text{every } X \in \mathcal{K} \text{ is nonsingular, with the prescribed signs at orders } n-1 \text{ and } n.} \tag{1}$$

Order one is immediate from the entry bounds. For order two, normalise to nonnegative entries and two-minors; adjacent two-minors are monotone in the checker coordinates, and staircase support plus telescoping extends this to all two-minors. This already settles $n \leq 4$.

2. Choose a bad minor of minimal span

Role. Start the contradiction. Let k be the first failing order, so $3 \leq k \leq n-2$ and all lower orders are correct on $\mathcal{K}$. Locate an index gap for the identity of Step 3.

A strictly wrong sign survives a small move towards the midpoint, so we may work at $M \in \operatorname{ri}\mathcal{K}$ (every free entry strictly inside its bounds). *Solid* minors (consecutive rows and columns) are never bad: in checker coordinates $z_{ij} = (-1)^{i+j}X_{ij}$, every partial derivative of the signed solid determinant is a $(k-1)$-minor with one common sign, so the determinant is monotone and bounded by its values at $A^{\pm}$, which are ≥ 0. Hence a bad k-minor of minimal total span has a gap; after transposing, a column gap:

$$\boxed{J = \{a\} \cup S \cup \{c\}, \qquad |R| = k, \quad |S| = r = k-2, \qquad C = [a+1, c-1], \quad |C| \geq r+1.} \tag{2}$$

3. The bad minor forces a low-rank rectangle

Role. Turn the sign violation into a linear-algebraic constraint: the rectangle of intermediate columns has rank exactly $k-2$.

Fix $R' \subset R$ with $|R'| = k-1$ and write $\Delta(T) = \det M[R, T]$, $d(T) = \det M[R', T]$. For any omitted $b \in C \setminus S$, Laplace expansion gives the three-term identity

$$\boxed{\Delta(J)\, d(S \cup \{b\}) = \Delta((J \setminus \{a\}) \cup \{b\})\, d(S \cup \{a\}) + \Delta((J \setminus \{c\}) \cup \{b\})\, d(S \cup \{c\}).} \tag{3}$$

If $d(S \cup \{b\}) \neq 0$, divide: $\Delta(J)$ is a nonnegative combination (the ratios of $(k-1)$-minors are ≥ 0 by induction) of two k-minors of *smaller* span, which have the correct sign by minimality. This contradicts the choice of $\Delta(J)$. So every $(k-1)$-minor on $S \cup \{b\}$ vanishes, each omitted column lies in the span of S, and

$$\boxed{\operatorname{rank} M[R, C] = k-2, \quad S \text{ a column basis.}} \tag{4}$$

4. Make the rank bound permanent

Role. Step 3 holds at one matrix. Push the rank bound to every matrix of the box, including both vertices; this is what allows a single entry to move in Step 5.

Permanent zeros. Let f be multiaffine in the d free coordinates, $f \geq 0$ on $\mathcal{K}$, and $f(M) = 0$ at $M \in \operatorname{ri}\mathcal{K}$. Writing the free coordinates of M as $\theta_j \in (0,1)$,

$$f(M) = \sum_{\omega \in \{0,1\}^d} \Big(\prod_j \theta_j^{\omega_j} (1-\theta_j)^{1-\omega_j} \Big) f(V_\omega), \tag{5}$$

a sum of positive weights times nonnegative vertex values. It vanishes only if every vertex value is zero, and then $f \equiv 0$ on $\mathcal{K}$. Apply this to the signed $(k-1)$-minors of the rectangle: they are ≥ 0 by induction and vanish at M by Step 3. Therefore

$$\boxed{\operatorname{rank} X[R,C] \leq k-2 \qquad \text{for every } X \in \mathcal{K}.} \tag{6}$$

Equation (5) is the Bernstein representation of multidegree $(1,\dots,1)$; nonnegativity is supplied by induction, not by vertex enumeration.

5. Exclude free entries and fixed rectangles

Role. Show that the permanent rank bound is incompatible with the bad minor. This is the technical core.

Set $p = k-3$ and $\delta = \varepsilon_p\varepsilon_{p+1}\varepsilon_{p+2}\varepsilon_{p+3}$. *Signed Schur elimination:* pick an invertible $p \times p$ pivot $H = X[P,Q]$ and form the Schur complement; after row/column sign corrections and a possible row reversal, the normalised matrix $\mathcal{T}_X$ is nonsingular, has nonnegative entries and two-minors, and

$$\det \widehat{S}_X[I,J] = \frac{\det X[P \cup I, Q \cup J]}{\det H}, \qquad \operatorname{rank} X[U_0,V_0] = p + \operatorname{rank} S_X[U_0 \setminus P, V_0 \setminus Q]. \tag{7}$$

At a sign-regular vertex its three-minors have weak sign δ; the bad k-minor becomes a bad three-minor. Three low-order facts about such matrices do the rest: positive entries in a row have interval support; if all three-minors are ≤ 0, only the upper-right and lower-left entries can vanish; and a positive vanishing two-minor forces a rank-one corner reaching a matrix boundary, with at most one orientation possible.

Case A: the rectangle has a free entry. Moving that entry alone, with the rank bound (6) in force, forces an *essential* row or column (one whose deletion lowers the rank). If $\delta = -1$, the residual rectangle has rank $\leq p$ at both vertices, and completing a pivot produces a zero block of size at least $(p+2) \times (p+2)$ or $(p+3) \times (p+1)$ in a matrix allowed at most two zeros: impossible. If $\delta = +1$, eliminate at M: an essential row leaves the middle column of the target three-minor zero on two rows, and an essential column leaves a zero interior column that, by interval support, kills an outer column. Either way the target determinant vanishes.

Case B: every entry of the rectangle is fixed. Use one pivot inside $R \times S$ at A^-, A^+ and M. The three residual target rows and the columns $C \setminus Q$ share a fixed rank-one block containing two columns positive on all three rows. Together with the first and last target rows they give a positive vanishing two-minor at *both* vertices; the corner lemma selects the same orientation at both, reaching the same outer column $d \in \{a,c\}$. Lifting through the pivot,

$$\boxed{\operatorname{rank} A^{\pm}[R, S \cup \{d\}] \leq r.} \tag{8}$$

Every $(r+1)$-minor of this rectangle is affine in the single varying column d and vanishes at both vertices, hence at the midpoint $\frac{1}{2}(A^- + A^+) \in \operatorname{ri}\mathcal{K}$; Step 4 makes it vanish at M. Adjoining the

other outer column raises the rank by at most one, so $\operatorname{rank} M[R, J] \leq k - 1$ and $\det M[R, J] = 0$: contradiction.

6. Complete the induction

Role. Discharge the contradiction hypothesis and pass from the interior to the boundary.

Both rectangle types are excluded, so no bad k-minor occurs in $\operatorname{ri} \mathcal{K}$. A strictly wrong sign on the boundary would persist on moving slightly towards the midpoint, so none occurs there either. Induction gives all minor signs on $\mathcal{K}$; nonsingularity was proved in Step 1. This proves the theorem.

Sharpness. Four exact rational examples show that nonsingularity of the vertices cannot be dropped, that the two vertices must share one signature, that the ordinary vertices L, U do not suffice, and that the theorem covers a degenerate four-parameter box outside all earlier fixed-entry criteria.

References

[1] J. Garloff. *Criteria for sign regularity of sets of matrices.* Linear Algebra Appl. 44 (1982), 153–160.

[2] J. Garloff, M. Adm, J. Titi. *A survey of classes of matrices possessing the interval property and related properties.* Reliable Computing 22 (2016), 1–14, Conjecture 3.1.

[3] Open Problems in Numerical Linear Algebra, problem IV-01. https://github.com/ajt60gaibb/OpenProblemsInNLA/tree/main/intervals-and-absolute-value-equations/IV-01.