Two Vertices Certify a Whole Interval of Nonsingular Sign-Regular Matrices

A proof of the Garloff–Adm–Titi interval conjecture

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Abstract

A real square entrywise matrix box $[L, U]$ consists entirely of nonsingular sign-regular matrices with a prescribed signature $\varepsilon \in \{-1, 1\}^n$ if and only if its two checker vertices A^- and A^+ have that property. Zero minors of every proper order and fixed entries of both checker parities are allowed. This proves, in every dimension and for every signature, the general interval conjecture of Garloff, Adm and Titi, which goes back to Garloff's 1982 interval criteria. The proof has six steps. Starting from the two vertices, we first obtain nonsingularity and the minor signs of orders $1, 2, n-1, n$ on the whole box. We then argue by induction on the minor order: a first wrong-sign minor of minimal span forces, through a three-term determinantal identity, a rank-deficient rectangle of intermediate columns; a multiaffine interpolation argument makes this rank bound permanent throughout the box; independent variation of a free entry then produces an essential row or column, which signed Schur complements and rank-one corner structure exclude, while a fully fixed rectangle is excluded by a common corner at the two vertices and an affine midpoint argument. Four exact rational examples show that none of the hypotheses can be dropped.

1 Introduction

1.1 Why two vertices

Interval matrices $[L, U] = \{X : L \le X \le U\}$ are the basic model for a matrix whose entries are known only up to bounds, and the central question of interval analysis is which properties of a matrix can be read off from finitely many of its vertices. For sign regularity, the natural candidates are the two *checker vertices* A^- and A^+ in (1.2), obtained by taking the lower bound on one checkerboard colour and the upper bound on the other. Garloff proved in 1982 that these two vertices certify strict sign regularity of the whole box, and the same for weak sign regularity under a restriction on the positions of the fixed entries [10]. Whether the two checker vertices certify *nonsingular* sign regularity in general, with zero proper minors and fixed entries of both parities allowed, has been open since then; it was stated as a conjecture by Garloff, Adm and Titi [12] and is problem IV-01 of the *Open Problems in Numerical Linear Algebra* catalogue [18]. Over four decades, the conjecture was settled for the totally nonnegative signature and for a list of special signatures (Section 1.3); the general case remained open, and the best general tests available used up to 2^{m+n-1} vertices rather than two.

The present paper proves the conjecture in full generality. The answer is the exact certification principle one hopes for: two matrices suffice for the whole box, in every dimension and for every signature.

1.2 The main theorem

For increasing index sets I, J of equal cardinality, write $X[I, J]$ for the corresponding submatrix, with its rows and columns in increasing order. A real $n \times n$ matrix X is *sign regular with signature* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$

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if

$$\varepsilon_k \det X[I,J] \geq 0 \qquad (|I| = |J| = k,\ 1 \leq k \leq n). \tag{1.1}$$

Thus zero minors are allowed. Strict sign regularity requires strict inequalities in (1.1). Nonsingular sign regularity requires (1.1) and $\det X \neq 0$.

Let $L, U \in \mathbb{R}^{n\times n}$ satisfy $L \leq U$ entrywise. Define the *checker vertices* of the entrywise box $[L,U]$ by

$$A^-_{ij} = \begin{cases} L_{ij}, & i+j \text{ even}, \\ U_{ij}, & i+j \text{ odd}, \end{cases} \qquad A^+_{ij} = \begin{cases} U_{ij}, & i+j \text{ even}, \\ L_{ij}, & i+j \text{ odd}. \end{cases} \tag{1.2}$$

Theorem 1.1. *Let $n \geq 1$, let $\varepsilon \in \{-1,1\}^n$, and let $L, U \in \mathbb{R}^{n\times n}$ satisfy $L \leq U$. The following statements are equivalent:*

(i) Every $X \in [L,U]$ is nonsingular and sign regular with signature ε.

(ii) Both matrices A^- and A^+ in (1.2) are nonsingular and sign regular with signature ε.

The implication from (i) to (ii) follows because the two vertices belong to the box. The content of the theorem is the converse, with no restriction on the positions of fixed entries or vanishing proper minors. In particular, a fixed entry need not have the same checker parity as every other fixed entry.

1.3 Earlier interval results and partial answers

The two-vertex interval property for strictly sign-regular matrices is classical. Garloff [10] established this property and a weak-sign version under a restriction on fixed positions: all fixed entries must have even index sum, or all must have odd index sum. The latter includes boxes without fixed entries; see also [12, Theorem 3.1] and [5, Theorems 3.2–3.3].

Garloff's 1982 conjecture for nonsingular totally nonnegative matrices was proved by Adm and Garloff [1, Theorem 3.6]. Here total nonnegativity means that every minor is nonnegative. Their proof uses the Cauchon algorithm, and their Theorem 3.4 also preserves a common zero/nonzero pattern of all minors. Earlier special cases included almost totally positive matrices [11]. Related singular cases require additional conditions: see [1, Corollaries 3.7–3.8] and the rank and independence hypotheses in [6].

Adm and Garloff [2] proved the interval property for nonsingular almost strictly sign-regular matrices and for nonsingular tridiagonal sign-regular matrices of a common signature. They also treated totally nonpositive matrices and related signatures. Further signatures, including those of matrices with nonnegative proper minors and negative determinant, were treated in [3]; see [13, IP 2.8–2.9]. Hassuneh, Adm, and Garloff [14] considered matrices with nonpositive proper minors and positive determinant. Some of those results retain a strict condition on a corner entry. The resulting lists involve eight period-four signatures and their final-sign reversals. The survey [5] records the coverage of small orders together with the relevant strict-entry qualifications.

The general statement is Conjecture 3.1 of Garloff, Adm, and Titi [12, p. 7]: nonsingular sign-regular matrices with a fixed signature should possess the interval property. This general conjecture is distinct from the totally nonnegative conjecture already settled in [1].

There are general tests using more vertices. For an $m \times n$ box, write

$$A_c = \tfrac{1}{2}(L+U), \qquad \Delta = \tfrac{1}{2}(U-L).$$

Adm and Garloff [4, Theorem 3.1] proved that the box is sign regular with a prescribed signature if and only if all matrices

$$A_c - \operatorname{diag}(y)\Delta \operatorname{diag}(z), \qquad y \in \{-1,1\}^m, \quad z \in \{-1,1\}^n, \tag{1.3}$$

have that signature. There are at most 2^{m+n-1} distinct matrices in (1.3). Their argument uses variation diminution; for related single-vector characterizations and interval-hull results, see [7, 8]. The January 2026 survey [5] explicitly leaves open whether, in the square nonsingular case, the larger vertex family can be reduced to the two checker vertices. Theorem 1.1 gives that reduction.

Two sufficient conditions in the public September 2026 manuscripts of Holden are closer to the boundary cases addressed here. Theorem G of [16] assumes an acyclic directed graph of nonzero fixed entries and confines fixed zero entries to one checker parity. A positive diagonal scaling then reduces to the older fixed-parity

result. Theorem V of [15] assumes no fixed zero entries and requires every fully fixed adjacent 2×2 block to be nonsingular. It controls order-three minors in every dimension and proves the full conclusion at order five. The unconditional order-two reduction in [17], together with the inverse-positive reduction in [16], supplies useful starting points. We include proofs of both reductions below. The higher orders are treated without the additional hypotheses in those manuscripts.

1.4 The boundary obstruction

The main issue is compatibility with the box. A nonsingular sign-regular matrix can be approximated by strictly sign-regular matrices with the same signature. Approximating the two checker vertices separately, however, does not ensure that the approximants remain checker-comparable in a way that yields the original interval conclusion. This difficulty is described explicitly in [5].

The proof below treats zero minors directly. Its determinant identities, inverse-positive arguments, and rank factorizations are classical. The corner structure is related to the shadow property for totally nonnegative matrices; see [1, Lemma 1.2] and the background in [9, 19]. We use these ideas in a form that depends only on the low-order signs visible after Schur elimination and that handles both signs of the resulting third-order minors.

1.5 The six steps of the proof

The proof is by induction on the minor order and is organised in six steps, one per section. A *bad* k-minor means $\varepsilon_k \det X[I, J] < 0$; a zero proper minor is never a contradiction by itself. The contradiction is always with a nonzero determinant or with nonsingularity of a full matrix.

Step	Main conclusion	Why it matters
1 (§2)	Nonsingularity; initial signs	Orders $1, 2, n-1, n$ hold throughout the box; this starts the induction and supplies nonsingularity for the later Schur and corner arguments.
2 (§3)	A minimal-span bad minor	Solid minors already have the correct sign, so a first violation must omit an interior row or column.
3 (§4)	A rectangle of rank $k-2$	A three-term determinant identity forces the omitted interior columns into the span of $k-2$ selected columns.
4 (§5)	A permanent rank bound	Lower-order signed minors vanish at an interior point; multi-affinity makes these zeros persist throughout the box.
5 (§6)	Exclude both rectangle types	A free entry forces an essential row or column; a fixed rectangle gives a common rank-one corner. Each leads to a contradiction.
6 (§7)	Finish the induction	Interior and boundary violations are excluded; combined with nonsingularity this proves the theorem.

1.6 Proof technique: a third-order obstruction

The following symbolic configuration illustrates the mechanism before the general argument. It is a hypothetical obstruction under the induction assumptions, not a numerical counterexample to Theorem 1.1. Consider a relative-interior matrix in a 5×5 entrywise box:

$$M = \begin{pmatrix} a_1 & u_1 & s_1 & v_1 & c_1 \\ a_2 & u_2 & s_2 & v_2 & c_2 \\ a_3 & u_3 & s_3 & v_3 & c_3 \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{pmatrix} \in \mathrm{ri}[L, U]. \tag{1.4}$$

Assume that orders one and two have the prescribed signs throughout the box (Step 1). Suppose that a wrong-sign three-minor of minimal total span occurs on $R = \{1, 2, 3\}$ and $J = \{1, 3, 5\}$ (Step 2). Write

$$B_{\mathrm{bad}} = M[R, J] = \begin{pmatrix} a_1 & s_1 & c_1 \\ a_2 & s_2 & c_2 \\ a_3 & s_3 & c_3 \end{pmatrix} = [\, a \; s \; c \,], \qquad \varepsilon_3 \det B_{\mathrm{bad}} < 0. \tag{1.5}$$

No claim is made that every possible obstruction can be moved to these row positions by a permutation.

From a bad minor to a rank-one rectangle (Step 3). Remove the two outer column indices $1, 5$ and include all intermediate columns, including those omitted by J. The rectangle to study is

$$F = M[R, \{2,3,4\}] = \begin{pmatrix} u_1 & s_1 & v_1 \\ u_2 & s_2 & v_2 \\ u_3 & s_3 & v_3 \end{pmatrix} = [\, u \; s \; v \,]. \tag{1.6}$$

The three-term identity of Lemma 4.1, applied to the omitted columns 2 and 4, forces every two-minor on columns $\{2,3\}$ and $\{3,4\}$ to vanish: otherwise the bad determinant would be a nonnegative linear combination of smaller-span three-minors with the correct sign. Since $s \neq 0$ by (1.5), there are scalars α, β with

$$F = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \begin{pmatrix} \alpha & 1 & \beta \end{pmatrix}, \qquad \operatorname{rank} F = 1. \tag{1.7}$$

From one matrix to the whole box (Step 4). Every signed two-minor of this rectangle is nonnegative throughout $[L, U]$ and vanishes at the relative-interior point M. Lemma 5.1 makes each such zero permanent, so

$$\operatorname{rank} X[R, \{2,3,4\}] \leq 1 \qquad \text{for every } X \in [L, U]. \tag{1.8}$$

It is this local rank bound that passes to the whole box; neither the vectors in (1.7) nor the rank of the full matrix is asserted to stay fixed.

Why this contradicts the bad minor (Step 5). Here $k = 3$, so the pivot size $p = k - 3$ is zero and no Schur elimination is needed; for a general bad $k \times k$ submatrix one first eliminates a nonsingular $(k-3) \times (k-3)$ pivot and tracks the ordering signs (Section 6.4). If the rectangle has a free entry, independent variation of that entry and Lemma 6.6 force an essential row or column, and the low-order structure of Section 6.1 excludes the resulting configurations. If the rectangle is fixed, the common rank-one corner argument reaches the same outer column at the two checker vertices, and the affine midpoint argument gives, at M, either $\operatorname{rank}[\, a \; s \,] \leq 1$ or $\operatorname{rank}[\, s \; c \,] \leq 1$. Adjoining the remaining outer column yields

$$\operatorname{rank} B_{\text{bad}} \leq 2, \qquad \det B_{\text{bad}} = 0, \tag{1.9}$$

contradicting (1.5). Excluding both branches removes the hypothetical bad minor, and repeating the argument through the minor orders finishes the proof (Step 6).

1.7 What the proof adds

The proof treats fixed entries of both checker parities and weak proper-minor signs simultaneously; it does not rely on every proper minor being nonzero, and it needs no perturbation to the strict case. Its ingredients are classical: determinant identities, inverse-positive matrices, rank factorizations and the shadow-type corner structure of totally nonnegative matrices. The new element is the chain *strict sign violation* $\to$ *minimal-span rectangle* $\to$ *permanent rank deficiency* $\to$ *impossible low-order Schur configuration*, which is what allows zero minors to be handled directly. The permanent-zero step is, in Bernstein-polynomial language, the statement that a multiaffine polynomial that is nonnegative on a box and vanishes at an interior point has all Bernstein coefficients zero; we use this with nonnegativity supplied by induction, not by vertex enumeration.

2 Step 1: nonsingularity and the initial minor orders

Role in the proof. Starting only from the two vertex hypotheses, we establish nonsingularity throughout the box and the prescribed signs at orders $1, 2, n-1, n$. This both starts the induction and supplies the nonsingularity needed for the later Schur and corner arguments. No bad minor is assumed in this step. We first fix notation.

2.1 Notation

All matrix indices start at 1. Index sets are always written in increasing order when a determinant is taken. For $a \le b$, write $[a,b] = \{a, a+1, \dots, b\}$ when an integer index set is intended, and $[n] = [1,n]$. Complements of index sets are taken in the full row or column index set of the matrix under discussion. We put $\varepsilon_0 = 1$ and give the empty determinant the value 1.

Let

$$D_n = \operatorname{diag}((-1)^1, \dots, (-1)^n).$$

The *checker order* is defined by

$$A \le_{\mathrm{c}} B \quad \Longleftrightarrow \quad D_n A D_n \le D_n B D_n.$$

For $A \le_{\mathrm{c}} B$, put

$$\mathcal{K}(A,B) = \{X : A \le_{\mathrm{c}} X \le_{\mathrm{c}} B\}.$$

This is an entrywise Cartesian box, whose ordinary lower and upper bounds need not be A and B. For the box in Theorem 1.1, they are the checker vertices $A = A^-$ and $B = A^+$. A coordinate is *fixed* if $A_{ij} = B_{ij}$ and *free* otherwise. A matrix lies in $\operatorname{ri}\mathcal{K}$ if each free coordinate lies strictly between its two bounds. Thus relative interior is taken in the affine space of the actual box, including when some or all coordinates are fixed.

The reductions below control orders $1, 2, n-1, n$. These exhaust the minor orders for $n \le 4$; at $n = 5$, order three is the first remaining order. The induction treats all subsequent orders by the same argument.

2.2 Nonsingularity and cofactor signs

Lemma 2.1. *Suppose $P, Q \in \mathbb{R}^{n\times n}$ are nonsingular, $P \le Q$, and $P^{-1}, Q^{-1} \ge 0$. Every C with $P \le C \le Q$ is nonsingular and satisfies $C^{-1} \ge 0$.*

Proof. Set

$$v = P^{-1}\mathbf{1}, \qquad w = Q^{-1}\mathbf{1}, \qquad T = Q^{-1}(Q - P).$$

Every row of an invertible nonnegative matrix has a positive entry, so v, w are strictly positive. Moreover, $T \ge 0$ and $Tv = v - w < v$. In the weighted maximum norm $\|x\|_v = \max_i |x_i|/v_i$, the norm of a nonnegative matrix satisfies

$$\|T\|_v = \max_i \frac{(Tv)_i}{v_i} < 1.$$

For $P \le C \le Q$, let $E = Q^{-1}(Q - C)$. Then $0 \le E \le T$, and hence $\|E\|_v < 1$. Since $C = Q(I - E)$, the Neumann series yields

$$C^{-1} = \left(\sum_{s=0}^{\infty} E^s\right) Q^{-1} \ge 0.$$

□

Corollary 2.2. *Suppose $A \le_{\mathrm{c}} B$ are nonsingular sign-regular matrices with signature ε. Every $X \in \mathcal{K}(A,B)$ is nonsingular, its determinant has sign ε_n, and its $(n-1)$-minors have their prescribed weak sign ε_{n-1}.*

Proof. The assertion about order zero is interpreted by the empty determinant when $n = 1$. Put $\delta = \varepsilon_{n-1}\varepsilon_n$. The inverse cofactor formula gives

$$\delta D_n A^{-1} D_n \ge 0, \qquad \delta D_n B^{-1} D_n \ge 0.$$

If $\delta = 1$, apply Lemma 2.1 between D_nAD_n and D_nBD_n. If $\delta = -1$, apply it between $-D_nBD_n$ and $-D_nAD_n$. Thus every $X \in \mathcal{K}$ is nonsingular and satisfies $\delta D_n X^{-1} D_n \ge 0$. Since $\mathcal{K}$ is connected, its nowhere-zero determinant has the sign of $\det A$, namely ε_n. Applying the inverse cofactor formula once more gives the signs of the $(n-1)$-minors. □

2.3 Staircase support

Lemma 2.3. *Let $Y \in \mathbb{R}^{N\times N}$ be nonsingular and TN_2. Every diagonal entry of Y is positive. A zero $Y_{ij} = 0$ with $i < j$ forces $Y_{rc} = 0$ for $r \le i$ and $c \ge j$. A zero below the diagonal forces all entries southwest of it to vanish. Consequently, the positive entries of every row and column form a nonempty integer interval containing its diagonal position. The left and right endpoints of the row intervals are nondecreasing with the row index.*

Proof. Nonsingularity supplies a permutation π such that $\prod_i Y_{i,\pi(i)} > 0$. If π has an adjacent inversion, $\pi(i) > \pi(i+1)$, the two-minor on these rows and columns gives

$$Y_{i,\pi(i+1)}Y_{i+1,\pi(i)} \ge Y_{i,\pi(i)}Y_{i+1,\pi(i+1)} > 0.$$

Swapping these two permutation values therefore preserves positivity of the selected product. Repeatedly removing adjacent inversions reaches the identity permutation, proving $Y_{ii} > 0$ for every i.

Suppose $Y_{ij} = 0$ with $i < j$. For $c > j$, the two-minor on rows i, j and columns j, c reads $-Y_{ic}Y_{jj} \ge 0$, so $Y_{ic} = 0$; the same conclusion for $c = j$ is the assumption. If $r < i$ and $c \ge j$, the minor on rows r, i and columns i, c now reads $-Y_{rc}Y_{ii} \ge 0$. This proves northeast zero propagation. Transposition proves the southwest assertion.

A row cannot have a zero between two of its positive entries: to the right of its positive diagonal such a zero would force the rest of the row to vanish, and to the left it would force all earlier entries to vanish. Thus its support is an interval containing the diagonal. If a lower row began to the left of an earlier row, the southwest propagation from a zero just before the earlier row's support would give a contradiction. If an earlier row ended to the right of a later row, northeast propagation from the later row gives the analogous contradiction. Hence both support endpoints are nondecreasing. The column assertion follows by transposition. □

We refer to these support properties as *staircase support*. They will also be applied after Schur elimination and a sign normalization.

2.4 The first two minor orders

Proposition 2.4. *Under the hypotheses of Corollary 2.2, every matrix in $\mathcal{K}(A, B)$ has the prescribed signs at orders one and two.*

Proof. Order one follows entry by entry. To treat order two, negate all matrices if necessary so that their entries are nonnegative. Reversing all rows, if necessary, then makes the endpoint two-minors nonnegative. Negation multiplies every k-minor by $(-1)^k$, and row reversal multiplies it by $(-1)^{k(k-1)/2}$. Both operations map a checker box to a checker box, possibly exchanging its two vertices. We may therefore work with nonnegative endpoints whose two-minors are nonnegative.

On consecutive rows and columns, the determinant $ad - bc$ is nondecreasing in a, d and nonincreasing in b, c. Its two extrema over this block of the box occur at the restrictions of the two checker vertices. All adjacent two-minors throughout the box are therefore nonnegative.

Take $X \in \operatorname{ri}\mathcal{K}$. Its positive support is the union of the endpoint supports. Each row's two endpoint support intervals contain its diagonal position. Their union is thus an interval; its left endpoint is the minimum of two nondecreasing functions of the row, and its right endpoint is their maximum. Hence this union also has staircase support.

Consider rows $i < j$ and columns $k < \ell$. If either $X_{i\ell}$ or X_{jk} is zero, the desired inequality

$$X_{ik}X_{j\ell} \ge X_{i\ell}X_{jk} \tag{2.1}$$

is immediate. Otherwise every entry of X in $[i, j] \times [k, \ell]$ is positive. Indeed, for each intermediate row, its left support endpoint is at most that of row j, hence at most k, and its right endpoint is at least that of row i, hence at least ℓ. Multiply the adjacent inequalities

$$X_{rc}X_{r+1,c+1} \ge X_{r,c+1}X_{r+1,c}, \qquad i \le r < j, \quad k \le c < \ell.$$

All factors that occur on both sides are positive and can be cancelled. The resulting inequality is (2.1). Continuity extends it from $\operatorname{ri}\mathcal{K}$ to $\mathcal{K}$. Undoing the normalizations proves the stated signs. □

3 Step 2: choose a bad minor of minimal span

Role in the proof. This is the start of the contradiction argument. If the conclusion fails, choose the smallest failing minor order k; Step 1 ensures $3 \le k \le n-2$, so every lower-order minor already has the prescribed sign throughout $\mathcal{K}$. We assume that a bad k-minor exists and, at a relative-interior matrix M, select one of minimal total row and column span. The purpose is to locate an index gap to which the determinant identity of Step 3 can be applied. We are not constructing an actual counterexample to the theorem.

Fix $k \ge 3$. For the remainder of Steps 2–5, suppose that the vertex hypotheses hold and that every minor of order at most $k-1$ has its prescribed weak sign throughout $\mathcal{K} = \mathcal{K}(A,B)$. For a nonempty index set I, its *span* is $\max I - \min I$.

Lemma 3.1. *Every solid k-minor has its prescribed sign throughout $\mathcal{K}$.*

Proof. Let the solid row and column intervals start at i_0, j_0. Use checker coordinates $z_{ij} = (-1)^{i+j} X_{ij}$, which increase from A to B. For an entry in local row u and column v of this block, the derivative of its signed determinant with respect to z_{ij} is

$$\varepsilon_k (-1)^{i+j+u+v} \det X[I \setminus \{i\}, J \setminus \{j\}].$$

Here $i = i_0 + u - 1$ and $j = j_0 + v - 1$, so the parity factor is the constant $(-1)^{i_0+j_0}$. By the induction hypothesis, every such derivative has weak sign $(-1)^{i_0+j_0}\varepsilon_k\varepsilon_{k-1}$, independent of the chosen entry. The signed determinant is thus monotone in one common checker direction in all its free coordinates. Its minimum is attained at A or at B, where it is nonnegative. □

For a nonempty index set I, its span is $\max I - \min I$.

Lemma 3.2 (Selection)**.** *If a wrong-sign k-minor occurs in $\mathcal{K}$, it occurs at a matrix $M \in \operatorname{ri}\mathcal{K}$. At that matrix choose a wrong-sign k-minor minimizing the sum of its row and column spans. After transposition if necessary, it can be written on rows R and columns*

$$J = \{a\} \cup S \cup \{c\}, \qquad |R| = k, \quad |S| = r = k-2, \qquad C = [a+1, c-1], \quad |C| \ge r+1. \tag{3.1}$$

The set C contains all columns between the two outer columns, including those omitted by J.

Proof. A strictly wrong signed value persists under a sufficiently small motion toward the midpoint of the box, so it occurs in $\operatorname{ri}\mathcal{K}$. There are finitely many minors, so a minimum-span choice exists. By Lemma 3.1, its rows and columns cannot both be consecutive. Transposition preserves the hypotheses and the sum of the spans; we may arrange that J has a gap. Then $|C| \ge k-1 = r+1$. □

4 Step 3: the bad minor forces a low-rank rectangle

Role in the proof. Continue under the contradiction hypothesis and the minimal-span choice of Step 2. We now prove a consequence of that hypothesis: the intermediate-column rectangle $M[R,C]$ has rank exactly $k-2$. The low-rank conclusion is derived, not introduced as another assumption. It converts a sign violation into a linear-algebraic constraint. The tool is a three-term determinantal identity of Fekete type.

4.1 A three-term determinantal identity

Lemma 4.1. *Fix a k-element row set R and a $(k-1)$-element subset $R' \subset R$. Write a k-element column set as $J = \{a\} \cup S \cup \{c\}$, where $a = \min J$, $c = \max J$, and $|S| = k-2$. Let $a < b < c$ with $b \notin J$. For increasing column sets define*

$$\Delta(T) = \det Z[R,T], \qquad d(T) = \det Z[R',T]$$

when the displayed determinants have the appropriate orders. Then

$$\begin{aligned}\Delta(J)d(S\cup\{b\}) = {}& \Delta((J\setminus\{a\})\cup\{b\})d(S\cup\{a\}) \\ & + \Delta((J\setminus\{c\})\cup\{b\})d(S\cup\{c\}).\end{aligned} \tag{4.1}$$

The identity has no rank hypothesis.

Proof. Let $T = J \cup \{b\} = \{t_1 < \cdots < t_{k+1}\}$ and let w_s be column t_s restricted to the rows R. Laplace expansion of a determinant with a repeated row gives the cofactor-kernel identity

$$\sum_{s=1}^{k+1} (-1)^s \Delta(T \setminus \{t_s\}) w_s = 0.$$

Restrict this vector identity to R'. Apply the linear functional that appends its argument after the $k-2$ fixed columns indexed by S and takes the determinant. The terms indexed by S vanish because a column is repeated. The other three terms are those of a, b, c.

For a, moving the appended column to the first position crosses $k-2$ columns; with its kernel sign the coefficient is $(-1)^{k-1}$. If b occupies position t in T, then exactly $k-t$ members of S are larger than b. Its coefficient is $(-1)^t(-1)^{k-t} = (-1)^k$. The column c is already appended in increasing order and has coefficient $(-1)^{k+1}$. Moving the b term to the other side gives (4.1). □

If the $(k-1)$-minors have a common weak sign and $d(S \cup \{b\}) \neq 0$, equation (4.1) expresses $\Delta(J)$ as a nonnegative linear combination of two k-minors with smaller column span. A vanishing denominator is the obstruction that will produce a rank bound.

Proposition 4.2 (The rectangle has rank $k-2$)**.** *Let M, R, J, S, C, r be as in Lemma 3.2. Then*

$$\operatorname{rank} M[R, C] = r = k - 2, \tag{4.2}$$

and the columns S are a basis of this rectangle.

Proof. The chosen determinant is nonzero, so the columns S restricted to R are independent. Suppose an omitted interior column $b \in C \setminus S$ increased their rank to $r + 1 = k - 1$. Some $(k-1)$-element row set $R' \subset R$ would then give $d(S \cup \{b\}) \neq 0$ in Lemma 4.1. The two other k-minors in (4.1) have the same rows and strictly smaller column span. They therefore have the correct sign by minimality. All three $(k-1)$-minors have the same weak sign, with a nonzero denominator, so the identity would give the correct sign for the chosen k-minor as well. This contradiction proves that every interior column is in the span of S, establishing (4.2). □

5 Step 4: make the rank bound permanent

Role in the proof. Step 3 gives low rank at one matrix M. Here we prove the same rank upper bound for the corresponding rectangle in every matrix of $\mathcal{K}$, including the two checker vertices. This is what permits independent entry variation in Step 5. We use two separate facts about each signed minor f: $f \geq 0$ on $\mathcal{K}$ by induction, and $f(M) = 0$ by the rank conclusion. Nonnegativity alone does not imply $f(M) = 0$.

5.1 Interior zeros of multiaffine minors

Lemma 5.1. *If a minor of order at most $k-1$ vanishes at a point of $\operatorname{ri}\mathcal{K}$, it vanishes throughout $\mathcal{K}$. Consequently, if a submatrix has rank $s \leq k-2$ at a relative-interior point, its rank is at most s at every point of $\mathcal{K}$.*

Proof. List the genuinely free scalar coordinates of $\mathcal{K}$ as $x_1, \ldots, x_d$, each varying in a nondegenerate interval $[a_q, b_q]$. At an interior coordinate let $\theta_q = (x_q - a_q)/(b_q - a_q) \in (0, 1)$. A determinant is affine in each scalar entry separately. Repeated interpolation therefore expresses its signed value f as

$$f(x) = \sum_{\omega \in \{0,1\}^d} \left(\prod_{q=1}^{d} \theta_q^{\omega_q} (1 - \theta_q)^{1-\omega_q} \right) f(v_\omega), \tag{5.1}$$

where v_ω are all ordinary vertices of the box. Every weight is strictly positive, and every signed vertex value is nonnegative by the induction hypothesis. If $f(x) = 0$, every vertex value must vanish. Interpolating again, now at any point of the box, shows $f \equiv 0$. If $d = 0$, the box is a singleton and the assertion is immediate.

For the rank assertion, all $(s+1)$-minors of the submatrix vanish at the chosen interior point. Their order is at most $k-1$, so each vanishes throughout the box. The rank is consequently at most s everywhere. If the submatrix has fewer than $s+1$ rows or columns, the same conclusion is automatic. □

In particular, the rank at a relative-interior point is the maximum rank of that submatrix over the box whenever this lemma applies. The lemma uses all vertices only as an interpolation identity; it does not add vertex hypotheses to Theorem 1.1.

Corollary 5.2. *With M, R, C, r as in Proposition 4.2,*

$$\operatorname{rank} X[R, C] \le r = k - 2 \qquad \text{for every } X \in \mathcal{K}. \tag{5.2}$$

Proof. Since $r = k - 2$, Lemma 5.1 applies to all $(r+1)$-minors of the rectangle. They vanish at M and are nonnegative throughout $\mathcal{K}$ by induction, hence vanish throughout the box. □

Connection with Bernstein polynomials. Equation (5.1) is the tensor-product Bernstein representation of a polynomial of multidegree $(1, \dots, 1)$ on the box. A nonnegative multiaffine polynomial that vanishes at an interior point has all of its Bernstein coefficients equal to zero. Here nonnegativity is supplied by the induction hypothesis rather than by checking all vertices algorithmically; the lemma does not assert that arbitrary polynomial nonnegativity on a box can be certified by two vertices.

6 Step 5: exclude free entries and fixed rectangles

Role in the proof. We now show that the permanent rank bound of Step 4 is incompatible with the bad minor assumed in Step 2. This is the technical core. Signed Schur elimination reduces the target k-minor to a three-minor. If the intermediate rectangle contains a free entry, independent variation forces an essential row or column and a forbidden low-order configuration. If every entry is fixed, a common rank-one corner reaches an outer column and forces the original bad determinant to vanish. Both branches close the same contradiction. We first record the low-order structure (§6.1) and the elimination facts (§6.4) that drive it, then the essential-line lemma (§6.7), and finally the exclusion argument (§6.8).

6.1 Low-order corner structure

In this section $Y \in \mathbb{R}^{N \times N}$ is nonsingular and TN_2, with $N \ge 3$. When a sign $\eta \in \{-1, 1\}$ is specified, all three-minors of Y have weak sign η.

6.2 A positive vanishing two-minor

Lemma 6.1. *Suppose $i < j$, $k < \ell$, all four entries of $Y[\{i, j\}, \{k, \ell\}]$ are positive, and its determinant is zero. If the three-minors have weak sign η, one of the following rectangles has rank one:*

$$\begin{aligned} \eta = -1: &\quad [1, j] \times [1, \ell] \quad \text{or} \quad [i, N] \times [k, N], \\ \eta = +1: &\quad [1, j] \times [k, N] \quad \text{or} \quad [i, N] \times [1, \ell]. \end{aligned}$$

For these fixed indices and this value of η, at most one of the two displayed orientations can occur in a nonsingular matrix.

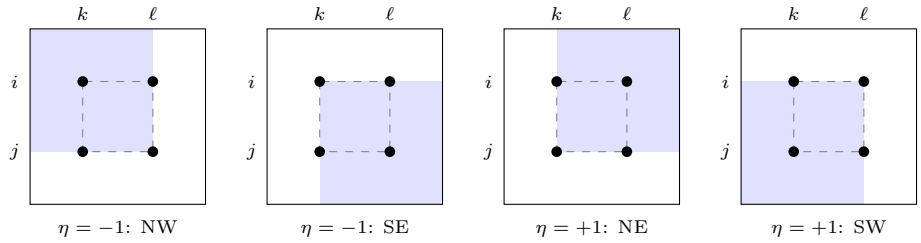


Figure 1: The four possible rank-one corner orientations. The dots mark the positive entries of the vanishing two-minor. Rows increase downward and columns to the right. These are schematic orientations; the rank bound in the proof excludes one member of the appropriate pair for any fixed set of indices.

Proof. Divide rows i and j by their positive column-k entries, and denote the normalized rows by u and v. There is a number $\rho > 0$ such that

$$u_k = v_k = 1, \qquad u_\ell = v_\ell = \rho.$$

Put $d_t = v_t - u_t$. The two-minors of these rows give

$$d_t \le 0 \ (t < k), \qquad d_t = 0 \ (k \le t \le \ell), \qquad d_t \ge 0 \ (t > \ell). \tag{6.1}$$

For example, inside $[k, \ell]$ the minors using column k give $d_t \ge 0$ and those using column ℓ give $d_t \le 0$. For each row h, put

$$q_h = Y_{h\ell} - \rho Y_{hk}.$$

Using the two-minors of columns k, ℓ and either anchor row gives

$$q_h \le 0 \ (h < i), \qquad q_h = 0 \ (i \le h \le j), \qquad q_h \ge 0 \ (h > j). \tag{6.2}$$

Some q_h is nonzero, because columns k, ℓ of the nonsingular matrix Y are linearly independent.

For $h < i$ or $h > j$ and $t < k$ or $t > \ell$, consider the three-minor on these three rows and columns, sorted in increasing order. After the two positive row normalizations, subtracting u from v shows that its value is

$$-d_t q_h. \tag{6.3}$$

For instance, in rows i, j, h and columns k, ℓ, t, the second row becomes $(0, 0, d_t)$, whose cofactor is $-q_h$. The other three placements give the same sign. The identity does not require entries of row h to be positive.

If both an earlier row with $q_h < 0$ and a later row with $q_h > 0$ existed, the common weak sign η in (6.3) would force $d_t = 0$ on both outer column ranges. Together with (6.1), this would make the two full rows proportional, contradicting nonsingularity. Thus the strict q_h occur on exactly one side of the anchor rows.

Suppose first that a later row $h > j$ has $q_h > 0$. Then $q_r = 0$ for every $r \le j$. Consider such a row with $Y_{rk} > 0$. Its normalized entries in columns k, ℓ are again $1, \rho$. For $r < j$, apply (6.3) to the pair of rows r, j and the later row h. If $\eta = +1$, the identity forces the normalized row difference to be nonpositive, while the two-minors make that difference nonnegative on columns $t > \ell$. It therefore vanishes there and throughout $[k, \ell]$. Thus row r is proportional to row j on all columns $t \ge k$. If $\eta = -1$, the same identity forces a nonnegative difference, which must vanish for $t < k$ and throughout $[k, \ell]$. In this case the rows are proportional on all columns $t \le \ell$. The conclusion for $r = j$ is immediate.

It remains to handle $r \le j$ with $Y_{rk} = 0$. Equation (6.2) gives $Y_{r\ell} = 0$. Staircase support and $Y_{jk} > 0$ imply $r < k$: a zero at or below the diagonal in column k could not have a positive entry farther down that column. Also $r < i$, since the column support contains every row between the two positive anchor entries. Northeast zero propagation now shows that row r is zero in every column $t \ge k$. When $\eta = +1$, this is precisely a zero row of the asserted northeast corner. When $\eta = -1$, it is impossible: on rows r, i, h and columns r, k, ℓ, the determinant is

$$Y_{rr} \det Y[\{i, h\}, \{k, \ell\}] = Y_{rr} Y_{ik} q_h > 0.$$

This contradicts the nonpositive third-order sign. Consequently the northeast corner has rank one for $\eta = +1$, and the northwest corner has rank one for $\eta = -1$. Its rank is exactly one because it contains the positive anchor entries.

If instead the strict q_h occur on the earlier side, reverse both rows and columns. A minor of order s acquires the factor $(-1)^{s(s-1)} = 1$, so its sign is preserved. The preceding argument then gives the southwest or southeast corner, respectively.

For uniqueness, any $p \times q$ submatrix of an invertible $N \times N$ matrix has rank at least $p + q - N$: adjoining the missing rows and columns can increase rank by at most $(N - p) + (N - q)$. A rank-one corner therefore has $p+q \le N+1$. For $\eta = -1$, a northwest corner would require $j+\ell \le N+1$, while a southeast corner would require $i + k \ge N + 1$. These inequalities cannot both hold because $i + k < j + \ell$. For $\eta = +1$, northeast and southwest would require $j \le k$ and $\ell \le i$, respectively, which are also incompatible. These necessary inequalities depend only on the anchor indices. They exclude different orientations even in two different nonsingular matrices with the same anchor indices and third-order sign. □

6.3 Zeros when the third-order sign is negative

Lemma 6.2. *If all three-minors of Y are nonpositive, its only possible zero entries are $(1, N)$ and $(N, 1)$.*

Proof. The diagonal is positive by Lemma 2.3. Suppose $Y_{ij} = 0$ with $i < j$. The rectangle $[1, i] \times [j, N]$ is zero. If $N - j + 1 \geq 2$, select two columns with indices at least j. They are independent because Y is nonsingular. They both vanish on the first i rows, so some two-minor on rows greater than i is nonzero, hence positive. Append row i and column i to this minor. The resulting sorted three-minor is Y_{ii} times the positive two-minor, contradicting the assumed sign. Hence $j = N$.

Similarly, if $i \geq 2$, two rows with indices at most i are independent and both vanish on columns $j, \ldots, N$. Some two-minor in columns less than j is therefore positive. Appending row j and column j gives another positive three-minor. Thus $i = 1$. Transposition treats zeros below the diagonal. □

Lemma 6.3. *A 3×3 TN_2 matrix having a zero in its middle column has nonnegative determinant.*

Proof. Write the matrix as $\left(\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right)$. If $e = 0$, two-minor nonnegativity gives $bd = fh = 0$. Expanding the determinant leaves $bfg + cdh \geq 0$. If $e > 0$ and $b = 0$, the upper-right two-minor gives $c = 0$, and the determinant is $a(ei - fh) \geq 0$. If $e > 0$ and $h = 0$, the lower-left two-minor gives $g = 0$, and the determinant is $i(ae - bd) \geq 0$. These cases cover every zero in the middle column. □

6.4 Signed Schur complements and rank completion

6.5 The ordering signs in Schur elimination

Let $Z \in \mathbb{R}^{n \times n}$ be nonsingular. Choose row and column sets P, Q with $|P| = |Q| = p$ such that $H = Z[P, Q]$ is nonsingular. The pivot need not be principal or contiguous. In the complementary index orders inherited from Z, set

$$S_Z = Z[P^c, Q^c] - Z[P^c, Q]H^{-1}Z[P, Q^c]. \tag{6.4}$$

For $p = 0$, the pivot is empty, its determinant is 1, and we interpret $S_Z = Z$. All subsequent formulas include this case.

For $i \notin P$ and $j \notin Q$, define

$$\alpha_i = (-1)^{|\{u \in P : u > i\}|}, \qquad \beta_j = (-1)^{|\{v \in Q : v > j\}|}. \tag{6.5}$$

Let $\widehat{S}_Z = \operatorname{diag}(\alpha) S_Z \operatorname{diag}(\beta)$.

Lemma 6.4. *For increasing sets $I \subseteq P^c$, $J \subseteq Q^c$ of equal cardinality,*

$$\det \widehat{S}_Z[I, J] = \frac{\det Z[P \cup I, Q \cup J]}{\det H}. \tag{6.6}$$

For any $U \supseteq P$ and $V \supseteq Q$, with no requirement that $|U| = |V|$,

$$\operatorname{rank} Z[U, V] = p + \operatorname{rank} S_Z[U \setminus P, V \setminus Q]. \tag{6.7}$$

In particular, S_Z is nonsingular.

Proof. First order the bordered rows by listing P followed by I, each internally increasing, and do the same with Q followed by J. Block elimination then gives the determinant $\det H \det S_Z[I, J]$. Sorting the combined row list has sign $\prod_{i \in I} \alpha_i$: the entry i crosses exactly the pivot rows larger than i. The column sorting sign is $\prod_{j \in J} \beta_j$. Thus

$$\det H \det S_Z[I, J] = \left(\prod_{i \in I} \alpha_i \right) \left(\prod_{j \in J} \beta_j \right) \det Z[P \cup I, Q \cup J].$$

Multiplying by these two sign products gives (6.6).

For (6.7), perform the same row and column permutations and invertible block elimination on $Z[U, V]$. The result is block diagonal with blocks H and $S_Z[U \setminus P, V \setminus Q]$. Their ranks add. Taking $U = V = [n]$ proves nonsingularity of S_Z. □

Suppose now that $\det H$ has sign ε_p, and that the relevant minors of orders $p+1$ and $p+2$ have their prescribed signs. Define

$$T_Z = (\varepsilon_p \varepsilon_{p+1}) \widehat{S}_Z. \tag{6.8}$$

Its entries are nonnegative. Equation (6.6) gives its order-two sign as

$$\tau_2 = \varepsilon_p \varepsilon_{p+2}.$$

If the order-$(p+3)$ signs are prescribed as well, its order-three sign is

$$\tau_3 = \varepsilon_{p+1} \varepsilon_{p+3}.$$

Indeed the scalar factor in (6.8) is raised to the minor order, and the denominator in (6.6) has sign ε_p.

If $\tau_2 = -1$, reverse all rows of T_Z. Row reversal multiplies both two-minors and three-minors by -1 and preserves entry signs. If $\tau_2 = +1$, leave the rows in place. We denote the resulting matrix by $\mathcal{T}_Z$. It is nonsingular and TN_2. Its prescribed third-order sign, when that order is known, is

$$\delta_p = \varepsilon_p \varepsilon_{p+1} \varepsilon_{p+2} \varepsilon_{p+3}. \tag{6.9}$$

The sign bookkeeping can be summarized as follows:

minor order	in T_Z	in $\mathcal{T}_Z$
1	$+1$	$+1$
2	$\varepsilon_p \varepsilon_{p+2}$	$+1$
3	$\varepsilon_{p+1} \varepsilon_{p+3}$	δ_p

After reversal, row indices and increasing row order refer to the new row positions; column order is unchanged.

Only the already established signs at orders $p+1$ and $p+2$ are needed to conclude that $\mathcal{T}_Z$ is TN_2. If an interior matrix has a wrong-sign bordered $(p+3)$-minor, the corresponding three-minor of $\mathcal{T}_Z$ has the wrong sign relative to (6.9). No third-order sign condition on this interior Schur matrix is being assumed. Further, the Schur matrices of checker-comparable matrices need not themselves be checker-comparable; the proof uses no such assertion.

6.6 Completion of a deficient pivot

Lemma 6.5. *Let $Z \in \mathbb{R}^{n \times n}$ be nonsingular and sign regular with signature ε. Let $0 \le p \le n-3$ and suppose $\delta_p = -1$ in (6.9). If*

$$\operatorname{rank} Z[R_0, C_0] = t \le p, \qquad |R_0| \ge p,$$

then

$$(|R_0| - p)(|C_0| - t) \le 2. \tag{6.10}$$

In particular, a rectangle of rank at most p cannot have either of the following sizes:

$$\text{at least } (p+2) \times (p+2), \qquad \text{at least } (p+3) \times (p+1). \tag{6.11}$$

Proof. Choose t columns in C_0 that form a basis of the columns restricted to R_0. The full rows R_0 of the nonsingular matrix Z are independent. Hence the matrix $Z[R_0, [n]]$ has rank $|R_0| \ge p$, and the selected t columns can be extended to p independent columns on R_0. Denote their index set by L. Every added column can be chosen outside C_0, because every column in C_0 already lies in the span of the initial t columns. Consequently,

$$|L| = p, \qquad |L \cap C_0| = t.$$

Choose p rows $P_0 \subseteq R_0$ for which $Z[P_0, L]$ is invertible. This is the pivot used for Schur elimination; it is formed by completing the basis when $t < p$, rather than assuming that the original rectangle has rank p.

For each column $c \in C_0$, its entries on all of R_0 are a linear combination of the initial t basis columns. Extending that coefficient vector by zeros in the other $p-t$ positions of L and restricting to P_0 shows that these

are exactly the coefficients computed by $Z[P_0, L]^{-1} Z[P_0, c]$. Thus its Schur residual is zero on $R_0 \setminus P_0$. The Schur matrix has a zero rectangle on

$$(R_0 \setminus P_0) \times (C_0 \setminus L)$$

of size $(|R_0| - p) \times (|C_0| - t)$.

The normalized Schur matrix has order $n - p \geq 3$, is nonsingular and TN_2, and has nonpositive three-minors. Diagonal sign changes and row reversal preserve zero entries. By Lemma 6.2, this whole Schur matrix has at most two zero entries. This proves (6.10). If the first size in (6.11) occurred, the zero rectangle would have at least 2×2 entries. For the second size it would have at least 3×1 entries. Both are impossible.

When $p = 0$, necessarily $t = 0$; the selected column and row sets are empty and no elimination is needed. The rectangle itself is zero in the normalized matrix, and exactly the same argument applies. This also verifies the empty-pivot case directly. □

6.7 Independent entry variation and essential lines

A row or column of a rank-r matrix is *essential* if deleting it reduces the rank to $r - 1$.

Lemma 6.6. *Let F be a rectangular matrix of rank $r \geq 1$. If $\operatorname{rank}(F + tE_{ij}) \leq r$ for all sufficiently small positive and negative t, then row i or column j of F is essential.*

Proof. Every r-minor after deletion of row i and column j must vanish. Otherwise, adjoining this row and column would give an $(r+1)$-minor of $F + tE_{ij}$ whose coefficient of t is, up to sign, that nonzero r-minor. Its constant term is zero because $\operatorname{rank} F = r$, so the rank bound would fail for nonzero small t.

Factor $F = UV$, where U has r independent columns and V has r independent rows. If neither indicated line were essential, then deleting row i from U and column j from V would leave both ranks equal to r. The first factor defines an injective map from $\mathbb{R}^r$, and the second a surjective map onto $\mathbb{R}^r$. Their product therefore has rank r. This is exactly the submatrix of F obtained by deleting both lines, in contradiction to the vanishing of all its r-minors. □

Apply this lemma to the rectangle of Proposition 4.2. If one of its entries is free, the relative-interior matrix M admits a small two-sided change in that entry alone. The permanent rank bound makes Lemma 6.6 applicable. After deleting the essential line, the residual rectangle has rank $r - 1$ at M. Its rank is at most $r - 1$ throughout $\mathcal{K}$ by Lemma 5.1, and in particular at both endpoints.

6.8 The exclusion argument

Proposition 6.7 (No bad minor in the interior)**.** *Let $3 \leq k \leq n$ and suppose that all minor orders through $k - 1$ have their prescribed weak signs throughout $\mathcal{K}$. Then no wrong-sign k-minor occurs at any matrix of $\operatorname{ri}\mathcal{K}$.*

Proof. Assume that a wrong-sign k-minor exists and take M, R, J, S, C, r from Lemma 3.2 and Proposition 4.2. Thus

$$J = \{a\} \cup S \cup \{c\}, \qquad C = [a + 1, c - 1], \qquad r = k - 2,$$

and the rectangle $M[R, C]$ has rank r, with at least $r + 1$ columns. Set

$$p = r - 1 = k - 3, \qquad \delta = \varepsilon_p \varepsilon_{p+1} \varepsilon_{p+2} \varepsilon_{p+3}.$$

We distinguish whether this interior-column rectangle is fixed throughout the box.

Case 1: the rectangle is fixed. Choose an invertible $p \times p$ pivot $H = M[P, Q]$ with $P \subset R$ and $Q \subset S$. Such a pivot exists because $M[R, S]$ has rank $r > p$; when $p = 0$, choose the empty pivot. The pivot and every entry in $R \times C$ are fixed throughout $\mathcal{K}$.

Using these same pivot positions, diagonal signs, and row reversal, form the normalized Schur matrices $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_M$ of Section 6.4. They are nonsingular and TN_2. The endpoints have weak third-order sign δ. The bad k-minor becomes a three-minor of $\mathcal{T}_M$ with the wrong sign relative to δ. Its columns are a, q, c, where q is the unique element of $S \setminus Q$, and its three rows are those of $R \setminus P$ in the order after any row reversal.

On these three rows and all columns of $C \setminus Q$, the three Schur matrices share one fixed rank-one block. Every term of the elimination formula for this block uses only entries in $R \times C$, so it is fixed. The rank identity (6.7)

gives rank $r - p = 1$. Column q is nonzero: applying the same rank identity to $M[R, S]$ shows that its single residual column has rank one. There are at least two columns in $C \setminus Q$, and the surviving columns a, c lie on opposite sides of all of them in the full Schur column order.

We claim that the shared block contains at least two columns positive on all three selected rows. If $\delta = -1$, no interior column of either endpoint Schur matrix can contain a zero, by Lemma 6.2: its only possible zero entries occur in the first and last columns. Thus every column of the block is positive on these rows.

If $\delta = +1$, the bad three-minor of $\mathcal{T}_M$ is negative. Lemma 6.3 forces its middle column q to be positive on all three selected rows. Every other nonzero column of the shared nonnegative rank-one block is a positive multiple of this column. If no second nonzero column existed, choose another column $b \in C \setminus Q$. It is zero on all three selected rows. Staircase support then forces these rows to be zero at a when $b < q$, or at c when $b > q$: in a row support interval, a zero lying to one side of a positive entry excludes every farther column on that side. Either conclusion would make the target three-minor zero, a contradiction. This proves the claim in both signs.

Choose column q and a second positive column of the fixed block. Together with the first and last of the three selected rows, they form a positive vanishing two-minor at both endpoints. Lemma 6.1 puts this anchor in a rank-one corner. The orientation is the same at A and B: its necessary index inequalities exclude one of the two possible orientations for both matrices. This corner contains all three selected rows and reaches one column boundary. It therefore contains the same outer column $d \in \{a, c\}$ at both endpoints, as well as column q.

Lift the rank bound through the common pivot. The submatrix on $R \times (S \cup \{d\})$ has just these three residual rows and the residual columns q, d. It follows that

$$\operatorname{rank} A[R, S \cup \{d\}] \leq p + 1 = r, \qquad \operatorname{rank} B[R, S \cup \{d\}] \leq r. \tag{6.12}$$

This endpoint statement must now be transferred to M. Every $(r+1)$-minor in (6.12) vanishes at both endpoints. All its columns except possibly d lie in the fixed set S. Hence its determinant is affine in the entries of its one remaining column d. It vanishes at the midpoint $(A + B)/2$. Minors omitting d are fixed and already zero.

The midpoint belongs to $\operatorname{ri} \mathcal{K}$, including when some entries are fixed. The order $r + 1 = k - 1$ is covered by the induction hypothesis. Lemma 5.1 therefore makes all these midpoint zeros permanent throughout $\mathcal{K}$, and in particular

$$\operatorname{rank} M[R, S \cup \{d\}] \leq r.$$

Adjoining the other outer column raises rank by at most one. The purported bad k-minor consequently has rank at most $r + 1 = k - 1$, contradicting its nonzero determinant. The affine midpoint step is essential here: endpoint zeros of an arbitrary multiaffine determinant alone do not imply an interior zero.

Case 2: the rectangle has a free entry and $\delta = -1$. Lemma 6.6 supplies an essential row or column. After its deletion, the rank is at most $r - 1 = p$ at each endpoint. Deleting a row leaves dimensions

$$(k - 1) \times |C| \;\geq\; (p + 2) \times (p + 2),$$

where the inequality compares both dimensions. Deleting a column leaves dimensions

$$k \times (|C| - 1) \;\geq\; (p + 3) \times (p + 1).$$

Both violate Lemma 6.5 at either nonsingular sign-regular endpoint. This case is impossible, even if the residual endpoint rank is strictly less than p.

Case 3: the rectangle has a free entry and $\delta = +1$. Again select an essential row or column. Schur elimination in this case is performed at M alone; its pivot is not required to be fixed or to remain invertible at the endpoints.

Suppose first that row i is essential. The columns S span the whole rectangle, so their restriction to $R \setminus \{i\}$ has rank $r - 1 = p$. Choose a nonsingular pivot on rows $P \subset R \setminus \{i\}$ and columns $Q \subset S$ of size p. On these remaining rows, every interior column lies in the span of the pivot columns. Its Schur residual is therefore zero outside the pivot rows. Among the three rows of the target three-minor, the two other than i are zero in its middle column, namely the unique column in $S \setminus Q$. The normalized Schur matrix is TN_2. Lemma 6.3 excludes the negative determinant that a wrong sign relative to $\delta = +1$ would require.

Suppose instead that column j is essential. Every column basis S of the rectangle contains j, since deletion of j lowers rank. Put $Q = S \setminus \{j\}$ and choose p rows $P \subset R$ giving an invertible pivot. The columns Q form a basis after deletion of j, so every other interior column has zero Schur residual on the three remaining selected rows. At least one such zero column b remains: $|C| \geq r + 1 = p + 2$, while only p pivot columns and column j have been excluded.

The target columns are a, j, c. A negative three-minor would force column j to be positive on all three selected rows by Lemma 6.3. The zero column b and staircase support then force column a to be zero on these rows if $b < j$, and column c to be zero if $b > j$. The target determinant is zero, a contradiction.

All three cases are excluded. Hence no wrong-sign k-minor occurs in $\operatorname{ri}\mathcal{K}$. □

7 Step 6: complete the induction

Role in the proof. We discharge the contradiction hypothesis. Since Step 5 excludes every possible intermediate rectangle, there can be no first failing minor order in the relative interior. We also explain why excluding interior violations excludes boundary violations. Together with Step 1, this yields nonsingular sign regularity on the whole box.

Proof of Theorem 1.1. Only (ii)⇒(i) needs proof. The case $n = 1$ is immediate, and Corollary 2.2 proves nonsingularity throughout the box for every n. Proposition 2.4 supplies the first two minor orders. Suppose, inductively, that all minor orders through $k-1$ are correct throughout $\mathcal{K}$, where $3 \leq k \leq n$. By Proposition 6.7, no wrong-sign k-minor occurs in $\operatorname{ri}\mathcal{K}$. A strictly wrong value at a boundary point would persist toward the midpoint, so none occurs on the boundary either. This completes the induction. All prescribed minor signs hold throughout $\mathcal{K}$, and its nonsingularity was proved separately in Corollary 2.2. □

The crucial chain is a strict sign violation, a minimal-span rectangle, permanent rank deficiency, and finally an impossible low-order Schur configuration. The affine midpoint argument in Case 1 of Proposition 6.7 is indispensable: zeros at two vertices of a general multiaffine polynomial do not force an interior zero, and the argument works only because all but one column of the relevant rectangle are fixed.

8 Exact examples

The first three examples isolate the roles of nonsingularity, the common signature, and the choice of vertices. The last gives a four-parameter degenerate box outside the preceding fixed-entry criteria. All entries and all checks in this section are rational.

Example 8.1 (Singular endpoints)**.** Consider

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Both are nonnegative of rank one, so every minor of order at least two is zero and both matrices are totally nonnegative. They satisfy $A \leq_c B$, since every differing position has even index sum. At their midpoint $M = (A + B)/2$, however,

$$\det M[\{1,3\},\{1,3\}] = \det \begin{pmatrix} 1/2 & 1 \\ 1/2 & 1/2 \end{pmatrix} = -\frac{1}{4}.$$

Thus singular checker vertices need not even preserve the weak minor-sign conclusion. For every $n \geq 3$, replacing A, B by $A \oplus I_{n-3}, B \oplus I_{n-3}$ gives the same failure in dimension n. A block-diagonal direct sum of totally nonnegative matrices is totally nonnegative: each nonzero minor is a product of minors in corresponding diagonal blocks, without an index permutation. The displayed bad minor persists in the first block.

Example 8.2 (A common signature is necessary)**.** The matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

are checker-comparable and strictly sign regular. Their signatures are $(+1,-1)$ and $(+1,+1)$, respectively, because their entries are positive and their determinants are -3 and 3. Their midpoint has every entry equal to $3/2$ and is singular. This is an example in dimension two of the all-dimensional theorem. Unlike total nonnegativity, arbitrary sign regularity with a prescribed signature is not automatically preserved by adjoining an identity block, so no such lifting is used here.

Example 8.3 (Ordinary interval bounds)**.** Let

$$L = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad U = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}.$$

These are ordinary entrywise lower and upper bounds, and both are strictly totally positive: their determinants are 1 and 3. Nevertheless, the permitted matrix

$$X = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \in [L, U]$$

has determinant -4. The ordinary lower and upper bounds cannot replace the checker vertices in Theorem 1.1. For $n \geq 2$, the corresponding direct sums with I_{n-2} give nonsingular totally nonnegative lower and upper bounds with the same bad two-minor at an interior or boundary choice.

Example 8.4 (A degenerate four-parameter box)**.** Put

$$(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(1, \frac{1}{10}, \frac{1}{100}, -\frac{1}{1000}, -\frac{1}{10000}\right), \qquad q = \frac{500618646037}{19352067067828800},$$

and define a rational 5×5 matrix by

$$W_{ij} = \sum_{s=0}^{4} \lambda_s \left(\frac{ij}{36}\right)^s - q\, \mathbf{1}_{\{i=j=1\}}, \qquad 1 \leq i, j \leq 5. \tag{8.1}$$

This seed appears in the example in [16, Section 6]. It is nonsingular and its signature is

$$(+1, +1, +1, -1, +1).$$

Its only zero minor is the order-three minor on rows and columns $\{1, 2, 3\}$.

With $D_5 = \operatorname{diag}(-1, 1, -1, 1, -1)$, let

$$H = -D_5 W^{-1} D_5, \qquad E = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A = H, \quad B = H + \tfrac{1}{8}E. \tag{8.2}$$

Then $A \leq_{\mathrm{c}} B$. The four entries in the northwest 2×2 block vary independently, each over an interval of width $1/8$; all other entries are fixed. Every entry of both endpoints is strictly positive. Exact evaluation gives the counts in Table 1, at *each* endpoint. The unique zero minor is

$$\det A[\{4,5\}, \{4,5\}] = \det B[\{4,5\}, \{4,5\}] = 0. \tag{8.3}$$

Thus both endpoints are nonsingular and have signature

$$\varepsilon = (+1, +1, -1, +1, -1).$$

Theorem 1.1 applies to the whole four-parameter box.

Here is why the earlier stated criteria do not cover this example. The fixed positions $(3,3)$ and $(3,4)$ have different checker parities, so the fixed-parity criterion does not apply. The directed graph in [16] has an edge $C_j \to R_i$ at a fixed positive entry of even index sum and an edge $R_i \to C_j$ at one of odd index sum. It contains the fixed-entry cycle

$$R_3 \longrightarrow C_2 \longrightarrow R_4 \longrightarrow C_3 \longrightarrow R_3.$$

Minor order	Positive	Zero	Negative
1	25	0	0
2	99	1	0
3	0	0	100
4	25	0	0
5	0	0	1

Table 1: Exact minor counts at each checker vertex in Example 8.4. There are 251 nonempty minors in each 5×5 matrix.

Thus Theorem G's acyclicity hypothesis fails. The adjacent 2×2 block in (8.3) is fully fixed and singular, so Theorem V of [15] also does not apply. The endpoints are dense, and the vanishing two-minor has four positive entries, so they are neither tridiagonal nor almost strictly sign regular. Finally, their first four signature signs $(+, +, -, +)$ do not occur in the eight period-four families listed in [14]; changing the final sign of a five-dimensional signature does not alter this exclusion.

The formulas (8.1)–(8.2) specify the matrices exactly without rounded entries. The accompanying standard-library rational verification program checks the seed, both checker vertices, the zero pattern, and all sixteen ordinary vertices of this box. At each of those sixteen vertices, every signed minor is nonnegative and the signed determinant is positive. Multiaffine interpolation therefore also gives an independent finite verification of this particular box. This finite calculation is an illustration; the general theorem is established by the preceding proof.

9 Scope and certification cost

Theorem 1.1 concerns real square boxes with independent scalar entry ranges and nonsingular endpoints of one prescribed signature. It treats every dimension, every such signature, and all allowable patterns of zero proper minors and fixed entries. The rationality of the examples imposes no restriction on the theorem. No perturbation that makes the proper minors strict is required.

The result includes the usual two-vertex interval conclusion for the nonsingular totally nonnegative, strictly sign-regular, and special-signature classes discussed in the introduction. It does not assert every stronger auxiliary conclusion of those papers, such as preservation of a whole zero/nonzero minor pattern. It also does not replace their results for singular or rectangular matrices under additional hypotheses. Example 8.1 shows that nonsingularity cannot be dropped from the general two-vertex statement.

The equivalence gives a reduction in the number of matrices whose nonsingular sign regularity must be certified: two checker vertices suffice for the entire square box. Checking that a given endpoint has the required signs is a separate recognition problem. An arithmetic or bit-complexity claim requires a specified endpoint-testing algorithm and a cost analysis; a measured computational speedup requires an implementation and comparison. Neither follows from the vertex count alone. The present result is the exact necessary-and-sufficient certification principle.

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Data availability. The exact rational checks of the examples in Section 8 and the verification scripts supporting the finite computations are available from the corresponding author on reasonable request.

Use of generative AI and AI-assisted technologies. During the preparation of this work, the authors used AI tools (Claude, Anthropic; Codex, OpenAI; Qwen APP, Alibaba) to generate candidate constructions and code, perform computations, draft mathematical arguments, and improve the readability of the text. After using these tools, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication. The theorem is established by the written proof; the four examples are checked in exact rational arithmetic.

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